# Alternating Current Network Theory

In general, if we have a sinusoidal signal (voltage, or current) applied to a circuit having at least one input and one output, we will expect a change in amplitude and phase at the output. The determination of these quantities for quite simple circuits can be very complex. It is indeed important to develop a convenient representation of sinusoidal signals to simplify the analysis of circuits in the AC regime.

## 10.1 Two-port Network

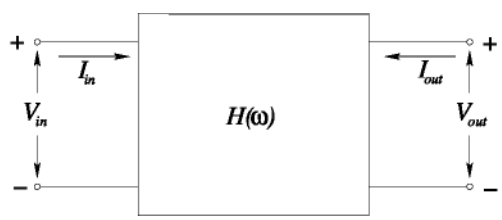


Figure 10.1: Two-port network circuit representation. The Voltage difference signs and current directions are conventional.

A linear circuit with one input and one output is called a two-port network (see figure 10.1).
To characterize the behavior of a two-port network, we can study the response of the output $V_o$ as a function of the angular frequency ω of a sinusoidal input $Vi$.
In general, we can write

$$V_o(\omega) = H(\omega)V_i(\omega), \text{ or } H(\omega) = \frac{Vo(\omega)}{Vi(\omega)}$$

where the complex function H(ω) is called the *transfer function of the two port network.* The transfer function contains the information of how the amplitude and the phase of the input changes when it reaches the output. Knowing the transfer function of a two-port network, we characterize completely the circuit. The definition of H(ω) suggests the way of measuring the transfer function. In fact, exciting the input with a sinusoidal wave we can measure the amplitude and the phase lead or lag respect to the input of the output signal.

To graphically represent H(ω), it is common practice to plot the magnitude |H(ω)| in a double logarithmic scale, and the phase arg [H(ω)] in a semilogarithmic scale for the angular frequency.

It is important to notice that it is not necessary to have an ideal sinusoidal generator to make the transfer function measurement. In fact, if the input amplitude changes with the frequency the ratio between the output will not change. The same is true for the phase, i.e. if the input phase changes the difference with the output phase cannot change.

Let's study two common two port networks, the RC low-pass filter and the RC high-pass filter circuits.

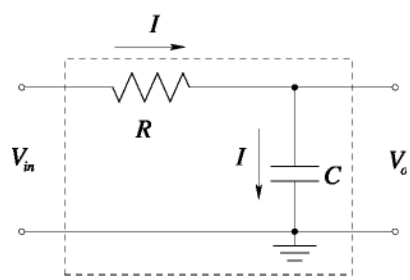


Figure 10.2: RC low-pass filter circuit.

### 10.2.1 The RC Low-Pass Filter

Figure 10.3 shows the *RC low-pass filter circuit.* The input and the output voltage differences are respectively ($V_{out}$ as function of $V_{in}$ can be directly calculated using the voltage divider equation)

$$V_{in} = Zi_{n}I = \left( R + \frac{1}{j\omega C} \right) I$$

$$V_{out} = Z_{out}I = \frac{1}{j\omega C} I$$

and the transfer function is indeed

$$H(\omega) = \frac{Vo_{ut}}{V_{in}} = \frac{1}{1 + j\tau\omega}, \qquad \tau\text{=RC}$$

or

$$H(\omega) = \frac{1}{1 + j\omega/\omega_o}, \quad \omega_o = \frac{1}{RC}$$

Computing the magnitude and phase of H(!), we obtain

$$|H(\omega)| = \frac{1}{\sqrt{1 + \tau^2\omega^2}}$$

$$\arg(H(\omega)) = -\arctan\left(\frac{\omega}{\omega_o}\right)$$

Figure 10.3 shows the magnitude and phase of H(ω). The parameter τ and $\omega_0$ are called respectively the *time constant* and the *angular cut-off frequency of the circuit.* The cut-off frequency is the frequency where the output $V_{out}$ is attenuated by a factor 1/√2.

It is worthwhile to analyze the qualitative behavior of the capacitor voltage difference $V_{out}$ at very low frequency and at very high frequency.

For very low frequency the capacitor is an open circuit and $V_{out}$ is essentially equal to $V_{in.}$ for high frequency the capacitor acts like a short circuit and $V_{out}$ goes to zero.

The capacitor produces also a delay as shown in the phase plot. At very low frequency the $V_{out}$ follows $V_{in}$ (they have the same phase). The output $V_{out}$ loses phase ( ωt = φ ) t = φ/ω) when the frequency increases the frequency $V_{out}$ starts lagging due to the negative phase ,φ, and then reaches a maximum delay due to a phase shift of -π/2.

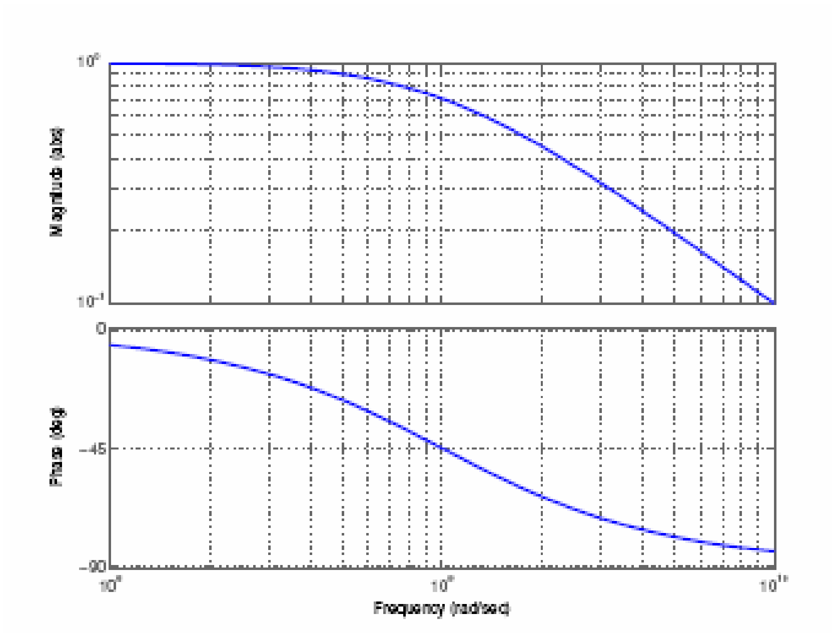


Figure 10.3: RC Low-pass filter circuit transfer funciton

### 10.4.2 The CR High-Pass Filter

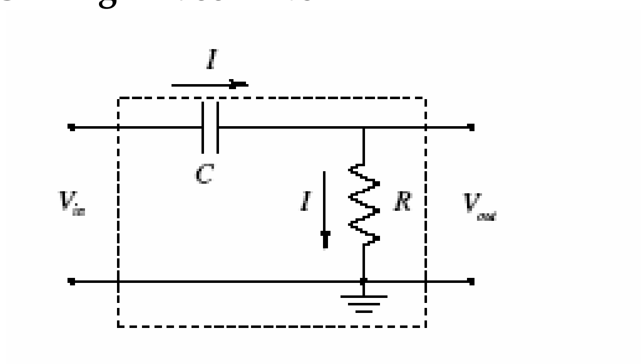


Figure 10.4: CR high pass filter circuit

Figure 10.4 shows the *CR high-pass filter circuit*. The input and the output voltage differences are respectively

$$V_{in} = Z_{in} I = \left( R + \frac{1}{j\omega C} \right) I$$

$$V_{out} = Z_{out} I = RI$$

and indeed the transfer function is

$$H(\omega) = \frac{V_{out}}{Vi_n} = \frac{j\omega\tau}{1 + j\tau\omega} \quad , \ \tau = RC$$

or

$$H(\omega) = \frac{j\omega / \omega_o}{1 + j\omega / \omega_o} \quad , \quad \omega_o = \frac{1}{RC}$$

Computing the magnitude and phase of H(ω), we obtain

$$|H(\omega)| = \frac{\tau\omega}{\sqrt{1 + \tau^2\omega^2}}$$

$$\arg(H(\omega)) = \arctan\frac{\omega_o}{\omega}$$

Figure 10.5 shows the magnitude and phase of H(ω). The definitions in the previous subsection for τ, and $\omega_0$ hold for the RC high-pass filter.

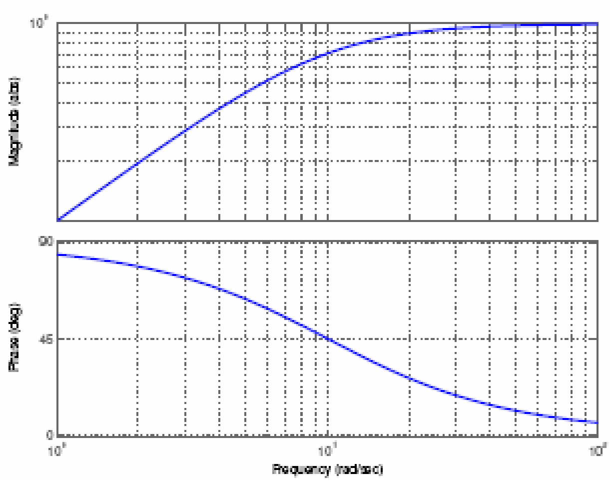


Figure 10.5: CR high pass filter circuit transfer function.