Semi-Direct Product of Edge Colored Graphs

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Abstract: A distance can be defined on a complete graph, called an edge colored graph. Edge colored graphs have many properties; however, this paper is mostly concerned with homogeneity and algebraic properties of edge colored graphs. This paper defines a semi-direct product of graphs, similar to a semi-direct product of groups. A semi-direct product of graphs is based upon similarities of the graphs that compose it. This paper also investigates the homogeneity of the semi-direct product, and the groups of isometries of semi-direct products in certain cases.

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Previous students have classified all edge colored graphs that are two-point homogeneous and have 12 or fewer vertices. Previous students have built graphs from smaller graphs using direct and wreath products. These products have helped students classify graphs, but they have not been enough. We define a new product of graphs, a semi-direct product for graphs, similar to the semi-direct product for groups. We prove that given necessary conditions this product is edge-homogeneous. We also describe and classify the isometries of certain products.

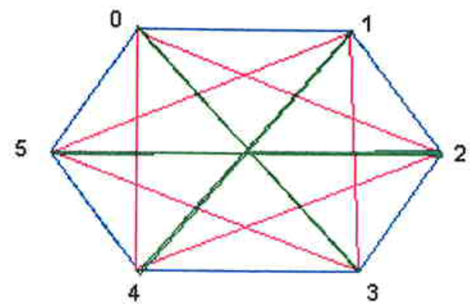
Definition: An edge colored graph is a complete graph G(V, E) with loops where a coloring function € maps the edges of a graph to a set of colors: For all vertices a, b $\in$ V, the edge {a, b} is mapped to a color €(a, b) defined by a coloring function €. Denote the set of colors for G(V, E) by €(G). The colors may be interpreted as distances in a graph.

Definition: The bijection $\alpha: V \to V$ is an isometry if and only if for all a, b $\in$ V, €(a, b) = €(α(a), α(b)). Denote the set of isometries for G(V,E) as I(G).

Definition: A graph G(V, E) is one-point homogeneous if and only if for all a, b $\in$ V, there exists an isometry $\alpha \in$ I(G) such that $\alpha(a) = b$.

Definition: An edge colored graph is two-point homogeneous when for all a, b, c, d in V, if €(a, b) = €(c, d) there exists an isometry $\alpha \in$ I(G) such that $\alpha(a) = c$ and $\alpha(b) = d$.

It is possible to build two-point homogeneous graphs from regular polygons, regular polyhedra and the Petersen graph. In general, a set of vertices in a metric space gives an edge colored graph by using the distance function as the color function. For example, a regular hexagon can be colored with four colors, as shown below (the fourth color is from a point to itself, and is the clear color in the picture). We call this graph C_6. The hexagon has 12 isometries, which form the group D_6. With this coloring scheme, C_6 is two-point homogeneous. For example, the edge connecting the points 0 and 1 can be mapped by rotation to the points 1 and 2. And a mirror reflection maps the points 0 and 1 to the points 2 and 1. In a similar way, any pair of points connected by one color can be mapped to any other pair of points connected by the same color.



C_6

Figure 1

Here are other examples of two-point homogeneous graphs.

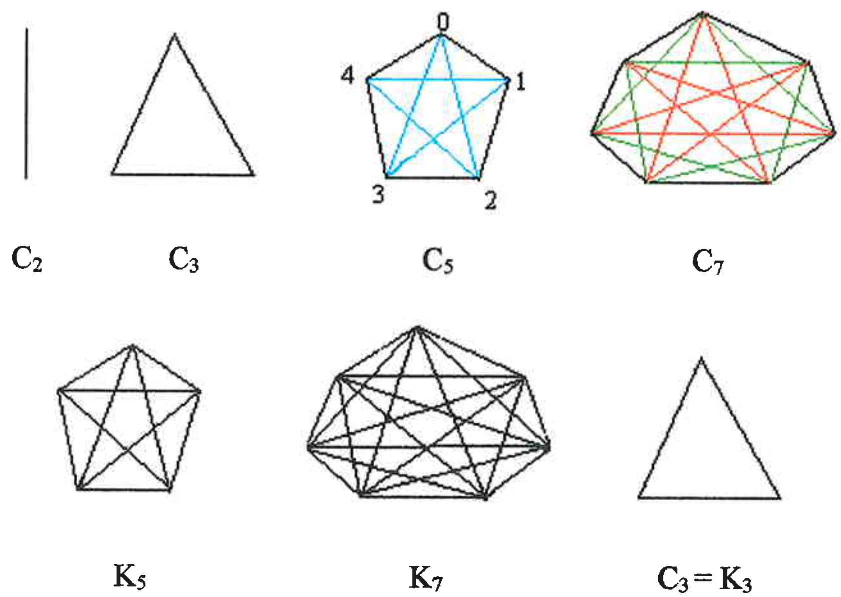


Figure 2

Definition: Let K_n denote the family of graphs with n vertices where the only two colors defined are one color from a point to itself (the clear color) and the other color is between any two distinct points.

Definition: Given a group H, define an "absolute value" on H as a function from H to H by $|a| = |b|$ if and only if $a = b$ or $b = a^{-1}$. Define $d(a, b) = |a*b^{-1}|$. A graph is derived from a group in the following way: K is the set of vertices and €(K) is the set of colors of K defined by $€(K) = \{|a|: a \in K\}$ where $€(a, b) = d(a, b) = |a*b^{-1}|$ and * is the operation on K.

In C_6, for example, we label the vertices with elements from Z_6. From this definition, €(0, 1) = €(0, 5) since 1 and 5 are inverses. €(4, 2) = |4 -2| = |2|. So, €(4, 2) = €(0, 2), since both sets of points are 2 away from each other. In a similar way, all the colors of C_6 are defined. Similarly, the family of graphs derived from the family of Z_n groups is denoted C_n.

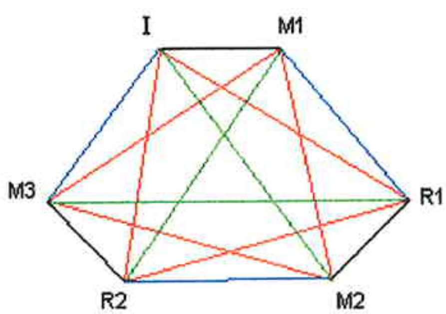
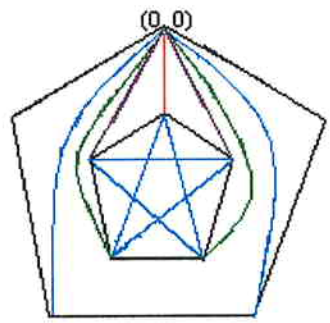


Figure 3

Another example is the graph shown above. This graph is derived from the group D_3. The red lines correspond to elements that are R1 apart (or R2 apart since R1 and R2 are inverses). The black line corresponds to elements M1 apart. Notice that d(I, M1) = d(R1, M2) = d(R2, M3). In the same way the blue lines correspond to elements M3 apart, and the green line to elements M2 apart. In this way, groups give rise to edge colored graphs.

Definition: A <u>direct product</u> of two graphs, denoted $A \times B$, is a complete graph on vertices $V = \{(a, b): a \in A, b \in B\}$ together with a coloring function €. The coloring function, $€(A \times B)$, is based on the coloring functions of A and B. The color from the point (a_1, b_1) to (a_2,b_2) is $\text{€}((a_1, b_1), (a_2, b_2)) = (\text{€}(a_1, a_2), \text{€}(b_1, b_2))$, where $\text{€}(a_1, a_2)$ is a color in $\text{€}(A)$ and $\text{€}(b_1, b_2)$ is a color in $\text{€}(B)$.

The colors of the new graph are based on the colors of the two original graphs, but they are new colors. Direct products may be used to build new graphs. For example the following is the direct product of C_2 with C_5, a graph with ten vertices.



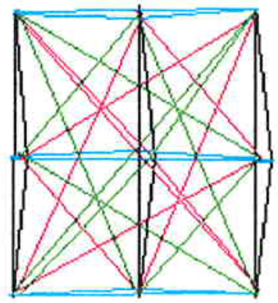
$C_2 \times C_5$

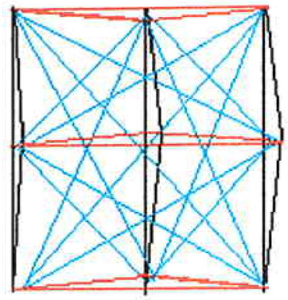
Figure 4

$C_2 \times C_5$ has two identical pentagons connected to each other in the same manner that (0, 0) is connected to the inner pentagon. $C_2 \times C_5$ is one-point homogeneous; which means all vertices look identical in $C_2 \times C_5$, even the points on the inner pentagon; or there exists an isometry that maps any point to any other point.

Definition: Let G, H be edge colored graphs. A <u>graph isomorphism</u> ψ:G $\rightarrow$ H is a one-to-one onto mapping that preserves color. That is, for all a, b, c, d $\in$ G if $\text{€}(a, b) = \text{€}(c, d)$ then $\text{€}(\psi(a), \psi(b)) = \text{€}(\psi(c), \psi(d))$ and if $\text{€}(a, b) \neq \text{€}(c, d)$ then $\text{€}(\psi(a), \psi(b)) \neq \text{€}(\psi(c), \psi(d))$. This means the same colors get mapped to the same color and different colors get mapped to different colors.

Note that $C_2 \times C_5$ is isomorphic to $C_5 \times C_2$. It is also isomorphic to C_{10}. However this is not always the case. For example $C_3 \times C_3$ is not isomorphic to C_9. $C_3 \times C_3$ (pictured below) has four colors. C_9 has five colors. Also, the direct product of graphs and groups do not correspond. For example, the graph derived from $Z_3 \oplus Z_3$ (pictured below) is not isomorphic to $C_3 \times C_3$, even though C_3 is derived from Z_3.





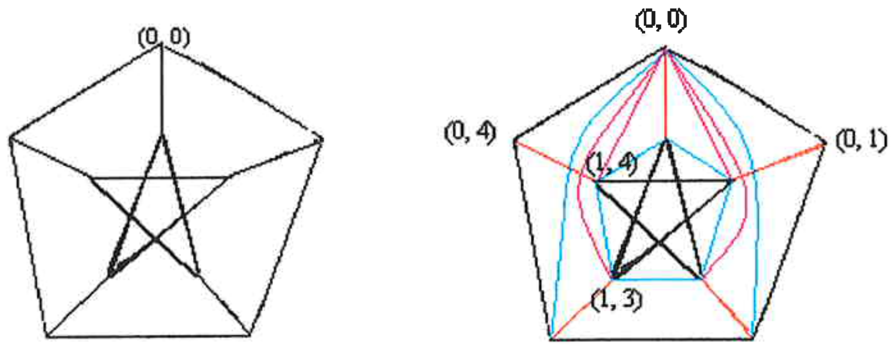
$Z_3 \oplus Z_3$ $\qquad$ $C_3 \times C_3$

Figure 5

With graphs of ten vertices or more, there exist graphs that are not in any previously described family of graphs, nor are they direct products of smaller graphs, nor are they derived from groups. The Petersen graph is the smallest such two-point homogeneous graph (Figure 6). However, it is a 'collapse' of the graph next to it, what I will call the pre-Petersen graph. By collapse, I mean that if certain colors of the pre-Petersen graph are grouped together and considered one color, the pre-Petersen graph would become the Petersen graph. The pre-Petersen graph is not two-point homogeneous. It is however 'edge homogeneous', which we now define.

Definition: A graph is edge homogeneous if it is one-point homogeneous and for all a, b, c, d $\in$ V if $\mathfrak{C}(a, b) = \mathfrak{C}(c, d)$ then there exists an isometry $\alpha \in I(G)$ such that either $\alpha(a) = c$ and $\alpha(b) = d$ or $\alpha(a) = d$ and $\alpha(b) = c$.

Edge homogeneity is weaker than two-point homogeneity. In the pre-Petersen graph the edge between (0, 0) and (1, 4) can only be mapped to the edge between (0, 0) and (1. 3) if (0, 0) is mapped to (1, 3) and (1, 4) is mapped to (0, 0). A necessary condition for a graph to be two-point homogeneous is the existence of an isometry that maps (0, 0) to (0, 0) and (1, 4) to (1, 3). However, there exists no such isometry for the pre-Petersen graph.



Petersen pre-Petersen

Figure 6

The Petersen graph has three colors: the color of the edges shown, the color from a point to itself and all other points are connected by the third color. In the pre-Petersen graph there exists an isometry that maps (0, 0) to all other vertices on the outer pentagon. To imagine the rest of the edges, just rotate (0, 0) around to all the vertices on the outer pentagon. I refrain from drawing in all the edges to keep that the graph readable.

Definition: A bijection φ on V is a <u>similarity</u> if and only if for all a, b, c, d $\in$ V if $\mathcal{E}(a, b) = \mathcal{E}(c, d)$ then $\mathcal{E}(\varphi(a), \varphi(b)) = \mathcal{E}(\varphi(c), \varphi(d))$. Define $\varphi^*: \mathcal{E}(G) \to \mathcal{E}(G)$ by $\varphi^*(\mathcal{E}(a, b)) = \mathcal{E}(\varphi(a), \varphi(b))$. Denote the similarities of a graph G(V, E) by S(G). In other words, a similarity is a permutation of the vertices that induces a permutation of the set of colors of a graph.

In the pre-Petersen graph the outside layer is a copy of C_5 while the inside layer is a copy of a similarity of C_5. The graph in figure 7 and 8 is another example of a product where the inside layer is a copy of a similarity of the other layers. The graph is split up, one picture depicting the colors within a layer, the other depicting the colors between layers. This split keeps the graph readable. To image the graph with all its colors, superimpose one over the other. This graph is composed from C_3 and C_7. Each layer looks like C_7. A similarity transforms each layer into the next layer. The second diagram depicts the colors between the layers.

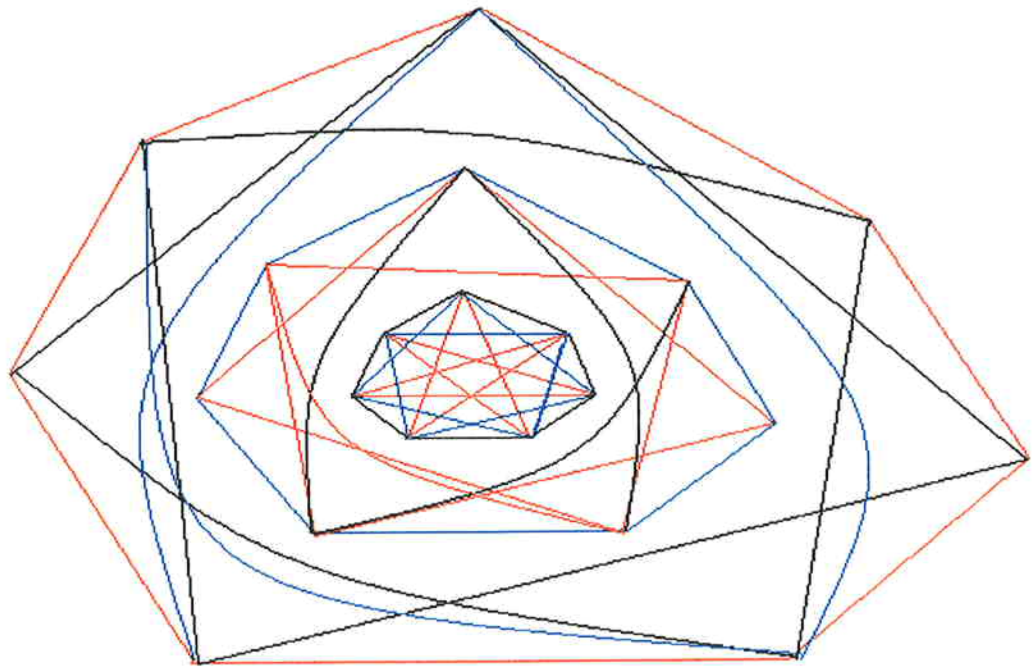


Figure 7

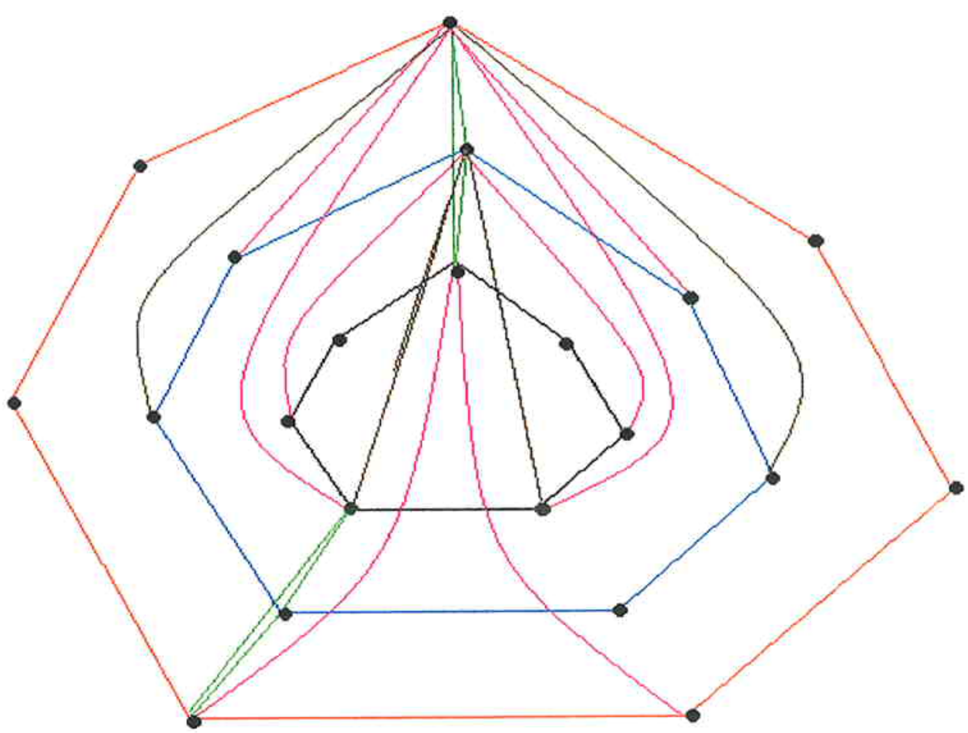


Figure 8

Notice that the inner most layer is one layer away from the outermost layer because of the structure of C_3. I have only drawn in the connecting colors from a few points. To fill in the rest of the edges rotate the graph in your mind.

The pre-Petersen graph and this 21-vertex graph are examples of 'semi-direct products' for graphs. I will now define the coloring scheme for a semi-direct product for graphs and describe all isometries of these graphs. First, I will describe the semi-direct product, denoted $\rhd_\varphi$, of C_2 with certain graphs. Then I will describe the semi-direct product of C_n $n > 2$ with certain graphs. I split these up into two categories because C_2 generates a different coloring scheme than C_n $n > 2$

Define $C_2 \rhd_\varphi B$

Definition: For two-point homogeneous graphs B, a semi-direct product, denoted $C_2 \rhd_\varphi B$, is defined when $\varphi \in S(B)$ and $\varphi \circ \varphi = \varphi^2 \in I(B)$. $C_2 \rhd_\varphi B$ is a complete graph with vertices $V = \{(a, b)\text{: } a \in C_2 \text{ and } b \in B\}$ together with a coloring function Є.

Before we define the coloring function, we define a set of equivalence classes of colors under φ.

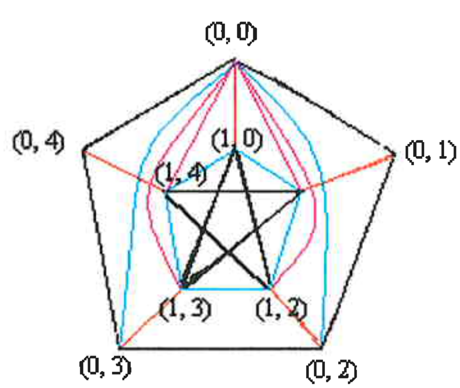
Definition: The set of equivalence classes under φ by Ç(B) = $\{[c_i]\text{: } c_i \in$ Є(B)$\}$ and $[c_i] = \{c_i, \varphi(c_i)\}$..

Since φ is a similarity it permutes colors; since φ^2 is an isometry, φ is a 2-cycle permutation, which means it switches two colors. Ç(B) is a set of equivalence classes where each equivalence class has a color c_i and the color with which φ permutes c_i, $\varphi(c_i)$. For example, C_5 has three colors (figure 2), the clear color and two other colors which φ permutes; then Ç(C_5) has two elements, the equivalence class consisting of the clear color and the equivalence class consisting of the other two colors.

Definition: The coloring function Є maps edges of $C_2 \rhd_\varphi B$ to colors in the following way:

$$\epsilon((a_1, b_1), (a_2, b_2)) = \begin{cases} (0, \epsilon(\varphi^k(b_1), \varphi^k(b_2))) & \text{if } a_1 = a_2 = k \\ (\epsilon(a_1, a_2), 0) & \text{if } a_1 \neq a_2 \text{ and } b_1 = b_2 \\ (\epsilon(a_1, a_2), \text{Ç}(b_1, b_2)) & \text{if } a_1 \neq a_2 \text{ and } b_1 \neq b_2 \end{cases}$$

With reference to the pre-Petersen graph, which is $C_2 \triangleright_\varphi C_5$, this coloring scheme is broken up into three components based on the stated conditions. The first component is coloring edges where the endpoints of the edge are in the same layer. Each layer is a transformation by a similarity of the previous layer. These are the colors of the B graph. The second and third components deal with coloring edges connecting layers. The second component consists of the non-clear color of C_2. Each point has exactly one edge of this color in the semi-direct product. In the pre-Petersen graph (figure 9), the second component is the red edge. The third component also describes how to connect layers: If $b_1 \neq b_2$, then colors are defined by equivalence classes. In the pre-Petersen graph (figure 9) the edge from (0, 0) to (1, 4) has the same color as the edge from (0, 0) to (1, 3) since in C_5, $\epsilon(0, 4) = \varphi(\epsilon(0, 3))$. We need to define the colors in terms of the equivalence classes in order for the graph to be edge homogenous. If these edges were not the same



pre-Petersen

Figure 9

color then the graph would not be one-point homogenous. For example, if we mapped (0,0) to (1, 0) by an isometry, (1, 0) is forced to go to (0, 0) since there is only one red edge connecting them. The (1, 4) would be get mapped to (0, 3) or (0, 2) since it is connected to (1, 0) by a blue edge. With out loss of generality, say it is mapped to (0, 3). Then (1,3) is forced to go to (0, 1) since it is connect to (1, 0) by a black edge and (1, 4) by a blue edge. Other points follow similarly and we obtain the picture below (figure 10). We can then rotate the graph to obtain the graph on the right in figure 10. Notice the edge from (0, 0) to (1, 3) is in the position that were originally occupied by (1, 4) to (0, 0). This example shows that if we want the semi-direct product to have even one-point homogeneity €((), 0), (1, 3)) = €((0, 0), (1, 4)). Therefore, the colors between layers depend upon equivalence classes of colors.

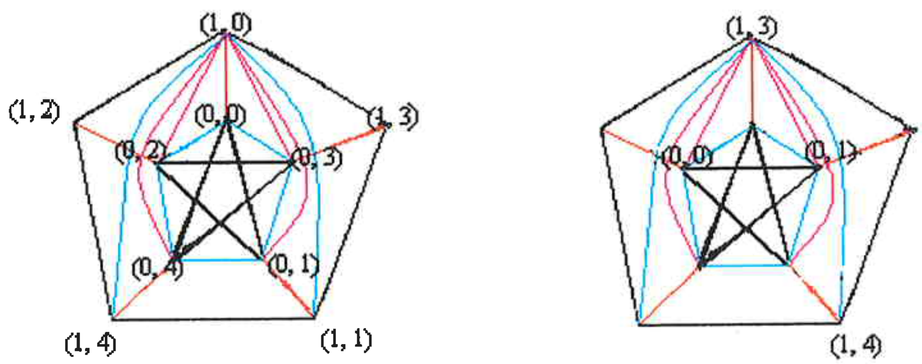


Figure 10

Also note, $\varphi^i \circ \varphi^j(a) = \varphi^{j+k}(a)$ does not necessarily equal $\varphi^{(j+k) \mathrm{mod}\ 2}(a)$, but €($\varphi^{j+k}(a), \varphi^{j+k}(b)$) = €($\varphi^{(j+k) \mathrm{mod}\ 2}(a), \varphi^{(j+k) \mathrm{mod}\ 2}(b)$). This means that φ^1 and φ^3 are both similarities and even permute colors the same way, but they may be different similarities because they may map the same vertex to different vertices.

Define $\gamma: C_2 \rhd_\varphi B \to C_2 \rhd_\varphi B$ by $\gamma(a, b) = (\alpha(a), \varphi^{-p}(b))$, where $\varphi^{-p} \in S(B)$ and α is a rotation in $I(C_2)$ given by $\alpha(a) = (a + p) \bmod 2 = m$, where $p \in Z$ and $m \in Z_2$. Where $m = 0$, γ maps points to the same layer. Otherwise, γ maps points to another layer.

Claim 1: γ is an isometry of $C_2 \rhd_\varphi B$.

I will divide the proof up into three cases, corresponding to the three components of the coloring function. Recall from the definition of an isometry, γ is an isometry if and only if $\text{€}(\gamma(a_1, b_1), \gamma(a_2, b_2)) = \text{€}((a_1, b_1), (a_2, b_2))$ in each case.

Case 1: $a_1 = a_2 = k$, so $\text{€}((a_1, b_1), (a_2, b_2)) = (0, \text{€}(\varphi^k(b_1), \varphi^k(b_2)))$.

$$
\begin{aligned}
\text{€}(\gamma(a_1, b_1), \gamma(a_2, b_2)) &= \text{€}((\alpha(a_1), \varphi^{-p}(b_1)), (\alpha(a_2), \varphi^{-p}(b_2))) \\
&= (\text{€}(\alpha(a_1), \alpha(a_2)), \text{€}(\varphi^m \circ \varphi^{-p}(b_1), \varphi^m \circ \varphi^{-p}(b_2))), \\
&\qquad \text{since } m = \alpha(a_1) = \alpha(a_2) = (k + p) \bmod 2, \\
&= (\text{€}(a_1, a_2), \text{€}(\varphi^{(k+p) \bmod 2} \circ \varphi^{-p}(b_1), \varphi^{(k+p) \bmod 2} \circ \varphi^{-p}(b_2))), \\
&\qquad \text{since } \alpha \text{ is an isometry}, \\
&= (\text{€}(a_1, a_2), \text{€}(\varphi^{(a+p)} \circ \varphi^{-p}(b_1), \varphi^{(a+p)} \circ \varphi^{-p}(b_2))), \\
&\qquad \text{by properties of } \varphi, \\
&= (\text{€}(a_1, a_2), \text{€}(\varphi^{a+p-p}(b_1), \varphi^{a+p-p}(b_2))) \\
&= (\text{€}(a_1, a_2), \text{€}(\varphi^{a}(b_1), \varphi^{a}(b_2))) \\
&= (\text{€}(a_1, a_2), \text{€}(\varphi^{k}(b_1), \varphi^{k}(b_2))), \\
&\qquad \text{since } a_1 = a_2 = k, \\
&= (0, \text{€}(\varphi^{k}(b_1), \varphi^{k}(b_2))), \\
&\qquad \text{since } a_1 = a_2, \\
&= \text{€}((a_1, b_1), (a_2, b_2)).
\end{aligned}
$$

Case 2: $a_1 \neq a_2$ and $b_1 = b_2$, so $\text{€}((a_1, b_1), (a_2, b_2)) = (\text{€}(a_1, a_2), 0)$.

$$
\begin{aligned}
\text{€}(\gamma(a_1, b_1), \gamma(a_2, b_2)) &= \text{€}((\alpha(a_1), \varphi^{-p}(b_1)), (\alpha(a_2), \varphi^{-p}(b_2))) \\
&= (\text{€}(\alpha(a_1), \alpha(a_2)), 0), \\
&\qquad \text{since } b_1 = b_2 \text{ implies } \varphi^{-p}(b_1) = \varphi^{-p}(b_2), \\
&= (\text{€}(a_1, a_2), 0), \\
&\qquad \text{since } \alpha \text{ is an isometry}, \\
&= \text{€}((a_1, b_1), (a_2, b_2)).
\end{aligned}
$$

Case 3: $a_1 \neq a_2$ and $b_1 \neq b_2$, so $\text{€}((a_1, b_1), (a_2, b_2)) = (\text{€}(a_1, a_2), \text{Ç}(b_1, b_2))$

$$
\begin{aligned}
\text{€}(\gamma(a_1, b_1), \gamma(a_2, b_2)) &= \text{€}((\alpha(a_1), \varphi^{-p}(b_1)), (\alpha(a_2), \varphi^{-p}(b_2))) \\
&= \text{€}((\alpha(a_1), \varphi^{-p \bmod 2}(b_1)), (\alpha(a_2), \varphi^{-p \bmod 2}(b_2)))
\end{aligned}
$$

$= \text{Є}((\alpha(a_1), \varphi^1(b_1)), (\alpha(a_2), \varphi^1(b_2)))$ or $\text{Є}((\alpha(a_1), b_1), (\alpha(a_2), b_2))$

But both of these options equal $(\text{Є}(\alpha(a_1), \alpha(a_2)), \text{Ç}(b_1, b_2))$,

$= \text{Є}((a_1, b_1), (a_2, b_2))$.

Hence, γ preserves color and consequently it is an isometry. ■

Next define $\delta: C_2 \triangleright_\varphi B \to C_2 \triangleright_\varphi B$ by $\delta(a, b) = (a, \beta(b))$, where $\beta \in I(B)$. Where γ maps points to another layer, δ maps points to other points on the same layer.

Claim 2: δ is an isometry of $C_2 \triangleright_\varphi B$.

Case 1: $a_1 = a_2 = k$, so $\text{Є}((a_1, b_1), (a_2, b_2)) = (0, \text{Є}(\varphi^k(b_1), \varphi^k(b_2)))$.

$\text{Є}(\delta(a_1, b_1), \delta(a_2, b_2)) = \text{Є}((a_1, \beta(b_1)), (a_2, \beta(b_2)))$

$= (\text{Є}(a_1, a_2), \text{Є}(\varphi^k \circ \beta(b_1), \varphi^k \circ \beta(b_2)))$,

since $a_1 = a_2 = k$

$= (\text{Є}(a_1, a_2), \text{Є}(\varphi^k(b_1), \varphi^k(b_2)))$,

since β is an isometry, it preserves color

$= \text{Є}((a_1, b_1), (a_2, b_2))$.

Case 2: $a_1 \neq a_2$ and $b_1 = b_2$, so $\text{Є}((a_1, b_1), (a_2, b_2)) = (\text{Є}(a_1, a_2), 0)$.

$\text{Є}(\delta(a_1, b_1), \delta(a_2, b_2)) = \text{Є}((a_1, \beta(b_1)), (a_2, \beta(b_2)))$

$= (\text{Є}(a_1, a_2), 0)$,

since $\beta(b_1) = \beta(b_2)$,

$= \text{Є}((a_1, b_1), (a_2, b_2))$.

Case 3: $a_1 \neq a_2$ and $b_1 \neq b_2$, so $\text{Є}((a_1, b_1), (a_2, b_2)) = (\text{Є}(a_1, a_2), \text{Ç}(b_1, b_2))$.

$\text{Є}(\delta(a_1, b_1), \delta(a_2, b_2)) = \text{Є}((a_1, \beta(b_1)), (a_2, \beta(b_2)))$

$= (\text{Є}(a_1, a_2), \text{Ç}(\beta(b_1), \beta(b_2)))$

$= (\text{Є}(a_1, a_2), \text{Ç}(b_1, b_2))$,

since β is an isometry,

$= \text{Є}((a_1, b_1), (a_2, b_2))$.

Hence, δ preserves color and consequently it is an isometry. ■

We have shown that γ and δ preserve color on $C_2 \rhd_\varphi B$ and are therefore isometries.

Define $C_n \rhd_\varphi B$

We now consider $C_n \rhd_\varphi B$, for $n > 2$. We consider the same bijections γ, δ and show they are isometries.

Definition: For a two-point homogeneous graphs B, a semi-direct product, denoted $C_n \rhd_\varphi B$, is defined when $\varphi \in S(B)$ and $\varphi^n \in I(B)$. $C_n \rhd_\varphi B$ is a complete graph with vertices $V = \{(a, b){:}\ a \in C_n \text{ and } b \in B\}$ together with a coloring function $\mathcal{E}$. The coloring function $\mathcal{E}$ maps edges of $C_n \rhd_\varphi B$ to colors in the following way:

$$\begin{aligned} \mathcal{E}((a_1, b_1), (a_2, b_2)) &= (0, \mathcal{E}(\varphi^k(b_1), \varphi^k(b_2))) && \text{if } a_1 = a_2 = k \\ &= (\mathcal{E}(a_1, a_2), 0) && \text{if } a_1 \neq a_2 \text{ and } b_1 = b_2 \\ &= (\mathcal{E}(a_1, a_2), \mathcal{E}(\varphi^k(b_1), \varphi^k(b_2))) \end{aligned}$$

if $a_1 \neq a_2$ and $b_1 \neq b_2$ where $|a_1 - a_2| = j$ and $i = n - j$, where $k = a_1$ if $j \leq i$ or $k = a_2$ if $i < j$.

Notice the first two components are the same as for $C_2 \rhd_\varphi B$. The third component arises from the structure of C_n. Recall that we derived C_n from the group Z_n. If $i < j$, a_1 and a_2 are i apart rather than j apart.

Also note, $\varphi^i \circ \varphi^j(a) = \varphi^{j+k}(a)$ does not necessarily equal $\varphi^{(j+k) \bmod n}(a)$, but $\mathcal{E}(\varphi^{j+k}(a), \varphi^{j+k}(b)) = \mathcal{E}(\varphi^{(j+k) \bmod n}(a), \varphi^{(j+k) \bmod n}(b))$. This means that φ^2 and φ^{2+n} are both similarities and even permute colors the same way, but they may be different similarities because they may map the same vertex to different vertices.

Define $\gamma{:}C_n \rhd_\varphi B \rightarrow C_n \rhd_\varphi B$ by $\gamma(a, b) = (\alpha(a), \varphi^{-p}(b))$, where α is a rotation in $I(C_n)$, $\varphi^{-p} \in S(B)$ and $\alpha(a) = (a + p) \bmod n = m$, where $p \in Z$ and $m \in Z_n$.

Claim 3: γ is an isometry of $C_n \triangleright_\varphi B$.

Case 1: $a_1 = a_2 = k$, so $\epsilon((a_1, b_1), (a_2, b_2)) = (0, \epsilon(\varphi^k(b_1), \varphi^k(b_2)))$.

This follows from case 1 in the proof for $C_2 \triangleright_\varphi B$ since the coloring is the same.

Case 2: $a_1 \neq a_2$ and $b_1 = b_2$, so $\epsilon((a_1, b_1), (a_2, b_2)) = (\epsilon(a_1, a_2), 0)$.

This follows from case 2 in the proof for $C_2 \triangleright_\varphi B$ since the coloring is the same.

Case 3: $a_1 \neq a_2$ and $b_1 \neq b_2$, so $\epsilon((a_1, b_1), (a_2, b_2)) = (\epsilon(a_1, a_2), \epsilon(\varphi^k(b_1), \varphi^k(b_2)))$.

Sub Case i: $j \leq i$, so $k = a_1$. Then $\alpha(a_1) = (k + p) \bmod n = m$.

$$
\begin{aligned}
\epsilon(\gamma(a_1, b_1), \gamma(a_2, b_2)) &= \epsilon((\alpha(a_1), \varphi^{-p}(b_1)), (\alpha(a_2), \varphi^{-p}(b_2))) \\
&= (\epsilon(\alpha(a_1), \alpha(a_2)), \epsilon(\varphi^m \circ \varphi^{-p}(b_1), \varphi^m \circ \varphi^{-p}(b_2))), \\
&\qquad \text{since } \alpha(a_1) = (k + p) \bmod n = m, \\
&= (\epsilon(a_1, a_2), \epsilon(\varphi^{(k+p) \bmod n} \circ \varphi^{-p}(b_1), \varphi^{(k+p) \bmod n} \circ \varphi^{-p}(b_2))), \\
&\qquad \text{since } \alpha \text{ is an isometry,} \\
&= (\epsilon(a_1, a_2), \epsilon(\varphi^{(k+p)} \circ \varphi^{-p}(b_1), \varphi^{(k+p)} \circ \varphi^{-p}(b_2))) \\
&= (\epsilon(a_1, a_2), \epsilon(\varphi^k(b_1), \varphi^k(b_2))) \\
&= \epsilon((a_1, b_1), (a_2, b_2)).
\end{aligned}
$$

Sub Case ii: $i < j$, so $k = a_2$. Then $\alpha(a_2) = (k + p) \bmod n = m$.

$$
\begin{aligned}
\epsilon(\gamma(a_1, b_1), \gamma(a_2, b_2)) &= \epsilon((\alpha(a_1), \varphi^{-p}(b_1)), (\alpha(a_2), \varphi^{-p}(b_2))) \\
&= (\epsilon(\alpha(a_1), \alpha(a_2)), \epsilon(\varphi^m \circ \varphi^{-p}(b_1), \varphi^m \circ \varphi^{-p}(b_2))), \\
&\qquad \text{since } \alpha(a_2) = (k + p) \bmod n = m, \\
&= (\epsilon(a_1, a_2), \epsilon(\varphi^{(k+p) \bmod n} \circ \varphi^{-p}(b_1), \varphi^{(k+p) \bmod n} \circ \varphi^{-p}(b_2))), \\
&\qquad \text{since } \alpha \text{ is an isometry.} \\
&= (\epsilon(a_1, a_2), \epsilon(\varphi^{(k+p)} \circ \varphi^{-p}(b_1), \varphi^{(k+p)} \circ \varphi^{-p}(b_2))) \\
&= (\epsilon(a_1, a_2), \epsilon(\varphi^k(b_1), \varphi^k(b_2))) \\
&= \epsilon((a_1, b_1), (a_2, b_2)).
\end{aligned}
$$

Hence, γ preserves color and consequently it is an isometry. ■

Now consider the bijection δ, as defined above, which maps points to other points on the same layer.

Claim 4: δ is an isometry of $C_n \triangleright_\varphi B$.

Case 1: $a_1 = a_2 = k$, so $\text{€}((a_1, b_1), (a_2, b_2)) = (0, \text{€}(\varphi^k(b_1), \varphi^k(b_2)))$.

Same as Case 1 for $C_2 \triangleright_\varphi B$ proof since the coloring is the same.

Case 2: $a_1 \neq a_2$ and $b_1 = b_2$, so $\text{€}((a_1, b_1), (a_2, b_2)) = (\text{€}(a_1, a_2), 0)$.

Same as Case 2 for $C_2 \triangleright_\varphi B$ proof.

Case 3: $a_1 \neq a_2$ and $b_1 \neq b_2$, so $\text{€}((a_1, b_1), (a_2, b_2)) = (\text{€}(a_1, a_2), \text{€}(\varphi^k(b_1), \varphi^k(b_2)))$.

Sub Case i: $j \beta i$, so $k = a_1$.

$$\begin{aligned}
\text{€}(\delta(a_1, b_1), \delta(a_2, b_2)) &= \text{€}((a_1, \beta(b_1)), (a_2, \beta(b_2))) \\
&= (\text{€}(a_1, a_2), \text{€}(\varphi^k \circ \beta(b_1), \varphi^k \circ \beta(b_2))), \\
&\qquad \text{since } a_1 = k, \\
&= (\text{€}(a_1, a_2), \text{€}(\varphi^k(b_1), \varphi^k(b_2))), \\
&\qquad \text{since } \beta \text{ is an isometry, it preserves color,} \\
&= \text{€}((a_1, b_1), (a_2, b_2)).
\end{aligned}$$

Sub Case ii: $i < j$, so $k = a_2$.

$$\begin{aligned}
\text{€}(\delta(a_1, b_1), \delta(a_2, b_2)) &= \text{€}((a_1, \beta(b_1)), (a_2, \beta(b_2))) \\
&= (\text{€}(a_1, a_2), \text{€}(\varphi^k \circ \beta(b_1), \varphi^k \circ \beta(b_2))), \\
&\qquad \text{since } a_2 = k \\
&= (\text{€}(a_1, a_2), \text{€}(\varphi^k(b_1), \varphi^k(b_2))), \\
&\qquad \text{since } \beta \text{ is an isometry, it preserves color,} \\
&= \text{€}((a_1, b_1), (a_2, b_2)).
\end{aligned}$$

Hence, δ preserves color and consequently it is an isometry. ■

Thus far, we have defined a semi-direct product for graphs and described some isometries of $C_n \triangleright_\varphi B$ $n \geq 2$. We will prove the semi-direct product of two graphs is edge-homogeneous.

Lemma 1: For a two-point homogeneous graph B, where $\varphi \in S(B)$ and $\varphi^n \in I(B)$ for $n \geq 2$. In $C_n \triangleright_\varphi B$ if $\epsilon((a, b), (x, y)) = \epsilon((a, b), (x, z))$, then there exists an isometry σ, such that $\sigma(a, b) = (a, b)$ and $\sigma(x, y) = (x, z)$.

Proof: Assume $(\epsilon(a, x), \epsilon(b, y)) = (\epsilon(a, x), \epsilon(b, z))$. So, $\epsilon(b, y) = \epsilon(b, z)$. Since B is two-point homogeneous there exists $\beta \in I(B)$ such that $\beta(b) = b$ and $\beta(y) = z$. Then let σ be defined by $\sigma(m, n) = (m, \beta(n))$, which is an isometry from above Claim 2 and 4. Then $\sigma(a, b) = (a, \beta(b)) = (a, b)$ and $\sigma(x, y) = (x, \beta(y)) = (x, z)$. Thus proves the lemma. ■

Theorem 1: For a two-point homogeneous graph B, where $\varphi \in S(B)$ and $\varphi^n \in I(B)$ for $n \geq 2$. $C_n \triangleright_\varphi B$ is edge homogeneous.

Proof: First, $C_n \triangleright_\varphi B$ is one-point homogeneous because γ maps a point to any layer and δ maps a point to any point within the same layer. So, under the composition of these two, a point (a, b) can be mapped to any other point in the graph. Since $C_n \triangleright_\varphi B$ is one-point homogeneous, it suffices to show that each edge from (0,0) can be mapped to all other edges of the same color from (0, 0). Also note that $\epsilon((a_1, b_1), (a_2, b_2)) = \epsilon((a_2, b_2), (a_1, b_1))$. I will split up the proof into three cases, each case corresponding to the three parts of the coloring function of the semi-direct product of graphs.

Assume $\epsilon((0, 0), (a, b)) = \epsilon((0, 0), (x, y))$. For each case I will show that there exist an isometry that maps the edge ((0, 0), (a, b)) to the edge ((0, 0), (x, y)). Since the graph is one-point homogeneous, this will prove the graph is edge homogeneous.

Case 1: $a = x = 0$, $b \neq 0$ and $y \neq 0$. This case corresponds to edges connecting (0, 0) to other vertices in the same layer. Since B is two point homogeneous, there exists

$\beta \in I(B)$, such that $\beta(0) = 0$ and $\beta(b) = y$. Then, let $\psi(u,v) = (u, \beta(v))$, which is an isometry by Claim 2 and 4. So, $\psi((0, 0)) = (0, \beta(0)) = (0, 0)$ and $\psi((a, b)) = (a, \beta(b)) = (x, y)$, since $a = 0$. So ψ maps $(0, 0)$ to $(0, 0)$ and (a, b) to (x, y). Therefore every edge from $(0, 0)$ to another vertex on the same layer can be mapped to every other edge the same color from $(0, 0)$.

Case 2: $a \neq 0$, $x \neq 0$ and $b = y = 0$. In the graph $C_2 \rhd_\varphi B$ there is only one edge from $(0, 0)$ with the color $(\epsilon(0, x), 0)$ and the identity maps this edge to itself. In $C_n \rhd_\varphi B$, for $n > 2$, there are at most 2 edges this color from $(0, 0)$, when $(a, b) \neq (x, y)$ but $a + x = n$, or $a = n - x$. Let $\sigma((0, 0)) = (\alpha(0), 0) = (x, y)$ where α is a rotation in C_n given by $\alpha(0) = (0 + p) \bmod n$. Then $p \bmod n = x$. Then $\sigma((a, b)) = (\alpha(a), b) = ((a + p) \bmod n, 0) = ((n - x + p) \bmod n, 0) = ((n \bmod n - x \bmod n + p \bmod n) \bmod n, 0) = ((0 - x + x) \bmod n, 0) = (0, 0)$. So σ maps $(0, 0)$ to (x, y) and (a, b) to $(0, 0)$. Therefore every edge from $(0, 0)$ to another vertex on a different layer (with the second component of the ordered pair being the clear color) can be mapped to every other edge the same color from $(0, 0)$.

Case 3: $0 \neq a$, $0 \neq b$, $0 \neq x$, and $0 \neq y$. If τ fixes $(0, 0)$ and maps (a, b) to (x, y) then we are done. So assume no such τ exists.

First, consider $C_2 \rhd_\varphi B$ only. Then $a = x = 1$. Also, let ψ map $(0, 0)$ (x, y) be given by $\psi((0, 0)) = (\alpha(0), \varphi^{-p}(0)) = (x, y)$, where $\varphi \in S(B)$ and p is an odd number, since α is a rotation such that $\alpha(0) =1$. We also assumed that $\epsilon((0, 0), (1, b)) = \epsilon((0, 0), (1, y))$, which means that $(\epsilon(0, 1), Ç(0, b)) = (\epsilon(1, 0), Ç(y, 0))$ and then $Ç(0, b) = Ç(y, 0)$. This in turn means that $\epsilon(0, b) = \epsilon(0, y)$ or $\epsilon(b, 0) = \epsilon(\varphi(0), \varphi(y))$ from the coloring definition. If the first is true then by the lemma there exists $\beta \in I(B)$ such that $\beta(0) = 0$ and $\beta(b) = y$ and so $\tau((0, 0)) = (0, 0)$ and $\tau((1, b)) = (1,\beta(b)) = (1,y)$. However, we assumed that no such τ existed, so $\epsilon(b,0) = \epsilon(\varphi(0), \varphi(y))$ must hold. We saw that $\varphi^{-p}(0) = y$. We want $\varphi^{-p}(b) = 0$, but it is conceivable that $\varphi^{-p}(b) = z$, where z is some other point not equal to 0. In fact, suppose $\varphi^{-p}(b) = z$ is true. Then $\epsilon(b, 0) = \epsilon(\varphi(z), \varphi(y))$ because φ^{-p} is a similarity and p is odd; then $\epsilon(\varphi(0), \varphi(y)) = \epsilon(\varphi(z), \varphi(y))$ and so $\epsilon(0, y) = \epsilon(z, y)$ because φ is a 2-cycle permutation of colors. Then from the lemma, there exists $\beta \in I(B)$ such that $\beta(z) = 0$ and $\beta(y) = y$ and so there exists σ such that $\sigma((0, z)) = (0, \beta(z)) = (0, 0)$

and $\sigma((1, y)) = (1, \beta(y)) = (1, y)$. Then $\sigma \circ \psi((0, 0)) = \sigma((1, y)) = (1, y)$ and $\sigma \circ \psi((1, b)) = \sigma((0, z)) = (0, 0)$. Then $\sigma \circ \psi$ maps $(0, 0)$ to (x, y) and (a, b) to $(0, 0)$.

Next consider $C_n \rhd_\varphi B$, for $n > 2$. Remember from the definition of colors that $j + i = n$. Say $|0 - a| = j$. If $|0 - x| = j$ then $a = j = x$ and let $\psi(u,v) = (u, \beta(v))$ where $\beta \in I(B)$ such that $\beta(0) = 0$ and $\beta(b) = y$. Then $\psi((0, 0)) = (0, 0)$ and $\psi((a, b)) = (x, y)$. So assume $|0 - x| = i$, then $a = n - i$. Let $\psi((0, 0)) = (\alpha(0), \varphi^{-i}(0)) = (x, y)$, where $\alpha(0) = 0 + i \bmod n = i$. Then $\alpha(a) = j + i \bmod n = n \bmod n = 0$. We want $\varphi^{-i}(b) = 0$, but it is conceivable that $\varphi^{-i}(b) = z$, for some $z \neq 0$. Assume $\varphi^{-i}(b) = z$; then $\psi((a, b)) = (\alpha(a), \varphi^{-i}(b)) = (0, z)$. The lemma guarantees that there exists a $\beta \in I(B)$ such that $\beta(z) = 0$ and $\beta(y) = y$. Then there exists σ such that $\sigma((0, z)) = (0, \beta(z)) = (0, 0)$ and $\sigma((x, y)) = (x, \beta(y)) = (x, y)$. So $\sigma \circ \psi((0, 0)) = \sigma((x, y)) = (x, y)$ and $\sigma \circ \psi((a, b)) = \sigma((0, z)) = (0, 0)$. Then $\sigma \circ \psi$ maps $(o, 0)$ to (x, y) and (a, b) to $(0, 0)$.

Similarly, let $|0 - a| = i$. If $|0 - x| = i$ then there is obviously an isometry that maps (a, b) to (x, y). So assume $|0 - x| = j$. Then $x = j = n - i = n - a$. Pick α as above so $\alpha(0) = x$ and $\alpha(a) = 0$. We want $\varphi^{-i}(b) = 0$, but it is conceivable that a z exists such that $\varphi^{-i}(b) = z$. Assume $\varphi^{-i}(b) = z$; since $(0, 0)$ and $(0, z)$ are on the same layer, by the lemma pick $\beta \in I(B)$ such that $\beta(z) = 0$ and $\beta(y) = y$. Then there exists σ such that $\sigma((0, z)) = (0, \beta(z)) = (0, 0)$ and $\sigma((x, y)) = (x, \beta(y)) = (x, y)$. So $\sigma \circ \psi((0, 0)) = \sigma((x, y)) = (x, y)$ and $\sigma \circ \psi((a, b)) = \sigma((0, z)) = (0, 0)$. Then $\sigma \circ \psi$ maps (a, b) to $(0, 0)$.

Therefore, $C_n \rhd_\varphi B$ is edge homogenous for all $n \geq 2$. ■

We have already described some isometries of a semi-direct product of two graphs. Next we will classify the group of isometries for certain products, namely $C_n \rhd_\varphi C_p$ for p prime and $n = (p-1)/2$.

Definition: Let K be a group and H be a subgroup of Aut(K). A <u>semi-direct product</u>, $H \rhd K$, is defined on $H \times K$ by $(h_1, k_1)(h_2, k_2) = (h_1 \circ h_2, k_1 \circ h_1(k_2))$ [1, p 99].

Theorem 2: For all odd primes p, $U(p) \rhd Z_p \approx S(C_p) \approx I(C_n \rhd_\varphi C_p)$, where $n = (p-1)/2$.

Proof: Let $s^i_a(x) = a^i x \bmod p$ and $r^j(x) = j + x \bmod p$ for some $i, j \in Z_p$, $a \in U(p)$ and $|a| = p-1$. Since s_a is a similarities and r is an isometries, the set generated by s_a and r, $<s_a, r>$, is a subgroup of $S(C_p)$. I will first show $<s^i_a, r^j>$ is isomorphic to $U(p) \rhd Z_p$, then I will count elements and show $<s^i_a, r^j>$ equals $S(C_p)$.

Let $\eta(s^i_a \circ r^j(x)) = (a^i, a^i j)$, where $a, i, j \in Z_p$, $a \in U(p)$, so $(a^i, a^i j) \in U(p) \rhd Z_p$. For 1 to 1, let $\eta(s^i_a \circ r^j(x)) = \eta(s^k_a \circ r^h(x))$. Then $(a^i, a^i j) = (a^k, a^k h) \Rightarrow a^i = a^k$ and $a^i j = a^k h \Rightarrow i = k$ and $j = h$. Thus, η is 1 to 1. For onto, let $(m, n) \in U(p) \rhd Z_p$. Since $U(p)$ is cyclic there exists an element $a \in U(p)$, of order $p-1$, and $i \in Z_p$ such that $a^i = m$. Since Z_p is a field there exist $j \in Z_p$ such that $a^i j = n$. For these i, j clearly $\eta(s^i_a \circ r^j(x)) = (a^i, a^i j) = (m, n)$. Therefore, η is onto. Before operation preservation, note how similarities and rotations can switch. Given $s^w_a \circ r^y(x)$, pick $z = a^w y$. $r^z \circ s^w_a(x) = z + a^w x \bmod p = a^w y + a^w x \bmod p = a^w(y + x) \bmod p = s^w_a \circ r^y(x)$. Now, for operation preserving: $\eta(s^i_a \circ r^j(x))\, \eta(s^k_a \circ r^h(x)) = (a^i, a^i j)(a^k, a^k h) = (a^{i+k}, a^{i+k}h + a^i j) = (a^{i+k}, a^{i+k}(h + j/a^k)) = \eta(s^{i+k}_a \circ r^{h+j/ak}(x)) = \eta(s^i_a \circ s^k_a \circ r^{j/ak} \circ r^h(x)) = \eta(s^i_a \circ r^j \circ s^k_a \circ r^h(x))$ because $z = a^k * j/a^k = j$, $w = k$ and $y = j$. So, η is operation preserving. Then, $<s^i_a, r^j> \approx U(p) \rhd Z_p$ and $|<s^i_a, r^j>| = |U(p) \rhd Z_p| = p(p-1)$.

In $S(C_p)$ for any i $|orb_{S(Cp)}(i)| = p$, since any point can be mapped by a similarity to any other point. Fix one point, 0. Each color in C_p has p edges because of the definition of coloring function for C_p and the fact that p is an odd prime. In addition, since similarities are permutations of colors, a similarity maps each color to another color. Therefore, a point adjacent to 0, call it 1, can be mapped by a similarity to any other point except 0. Once 0 and 1 are fixed, the color between them is determined. Next, consider the other point adjacent to 0, call it 2. Since similarities map colors to colors and $\mathcal{C}(0,1) = \mathcal{C}(0, 2)$, 2 is also fixed. In a similar way, all other points are fixed once two points are fixed. Refer to the graphs of C_5 and C_7 for examples. Consequently, $|stab_{S(Cp)}(i)| = p-1$. By the Orbit-Stabilizer Theorem, $|S(C_p)| = p(p-1)$. Then $<s^i_a, r^j>$ is a subgroup of $S(C_p)$ with the same number of elements, so $<s^i_a, r^j> = S(C_p)$. Then $U(p) \rhd Z_p \approx S(C_p)$. All we have left to show is that $S(C_p) \approx I(C_n \rhd_\varphi C_p)$.

By Claims 1-4 $\gamma(a, b) = (\alpha(a), \varphi^{-q}(b))$ and $\delta(a, b) = (a, \beta(b))$ are isometries of $C_n \rhd_\varphi C_p$ (recall $\varphi^{-q} \in S(C_p)$ and α is a rotation in $I(C_n)$ given by $\alpha(a) = (a + q) \bmod n = m$, where $p, m \in Z$ and $\beta \in I(C_p) = D_p$). Since the rotations are a subgroup of D_n, $\delta(a, b) = (a, \beta(b))$, where β is a rotation of C_p, is also an isometry. Then each $\gamma \circ \delta = (\alpha, \varphi^{-q} \circ \beta)$ is an isometry of $I(C_n \rhd_\varphi C_p)$.

Let $\eta(s^i_a \circ r^j(x)) = (\alpha, s^i_a \circ r^j) = \gamma \circ \delta$, where $\alpha(a) = (a + i) \bmod p$ and $s^i_a \circ r^j \in S(C_p)$ with s and r as above. For 1 to 1: let $\eta(s^i_a \circ r^j(x)) = \eta(s^k_a \circ r^l(x))$. Then $(\alpha_1, s^i_a \circ r^j) = (\alpha_2, s^k_a \circ r^h) \Rightarrow \alpha_1 = \alpha_2 \Rightarrow i = k$ and $j = h$. So, η is 1 to 1. For operation preservation: $\eta(s^i_a \circ r^j(x))\, \eta(s^k_a \circ r^h(x)) = (\alpha_1, s^i_a \circ r^j)(\alpha_2, s^k_a \circ r^h) = (\alpha, s^i_a \circ r^j \circ s^k_a \circ r^h)$, where $\alpha(a) = (a + i + k) \bmod p = \alpha_1(\alpha_2(x))$. And $\alpha_1(x) = (x + i) \bmod p$ and $\alpha_2(x) = (x + k) \bmod p$. Let $m = j/a^k$ or $j = ma^k$. Then $(\alpha, s^i_a \circ r^j \circ s^k_a \circ r^h) = (\alpha, s^i_a \circ s^k_a \circ r^{j/ak} \circ r^h) = \eta(s^{i+k}_a \circ r^{h+j/ak}(x)) = \eta(s^i_a \circ s^k_a \circ r^{j/ak} \circ r^h(x)) = \eta(s^i_a \circ r^j \circ s^k_a \circ r^h(x))$ because $z = a^k * j/a^k = j$, $w = k$ and $y = j$. Therefore, η preserves operation. For onto: since $C_n \rhd_\varphi C_p$ is one-point homogenous, $|orb_{I(C_n \rhd_\varphi C_p)}(i)| = pn = p(p-1)/2$. Fix the point (0, 0) and consider a point adjacent to it on the same layer, (0, 1). Since edges within a layer have different colors than edges that cross layers an isometry can only map (0, 1) to points within the 0 layer. And since each layer looks like C_p, there are only two isometries that fix (0, 0), the identity and a mirror reflection. Now consider a point adjacent to (0, 0) but on a different layer, (1, 0). If $n = 2$, there is only one edge of this color from (0, 0), so (1, 0) is fixed. If $n > 2$, there are only two edges this color from (0, 0) the edge ((0, 0), (1, 0)) and ((0, 0), (n-1, 0)). Nevertheless, it is impossible to map (1, 0) to (n-1, 0) while leaving (0, 0) fixed because our isometries $\gamma \circ \delta = (\alpha, \varphi^{-q} \circ \beta)$ consist of only rotations is the first coordinate, so if (0, 0) remains fixed, so does (1, 0). Similarly, all other points are also fixed and so the $|stab_{I(C_n \rhd_\varphi C_p)}(i)| = 2$. By the Orbit-Stabilizer Theorem $|I(C_n \rhd_\varphi C_p)| = p(p-1)$. Thus, η is onto because it is 1 to 1 and the two groups are equal in size. Therefore, $S(C_p) \approx I(C_n \rhd_\varphi C_p)$. Thus, the entire claim is true. ■

We have defined a semi-direct product for edge colored graphs and proved they are always edge homogenous. We have also classified the isometries for certain graphs. Classifying the isometries for all semi-direct products is a difficult task. The next easiest step would be to look at all graphs derived from finite fields, of which the C_p graphs are the most basic. We can represent the isometries of a C_p graph by one-dimensional affine transformations. In general, we can represent the isometries derived from the group $GF(p^n)$ by n dimensional affine transformations. However, the affine transformations do not completely classify all isometries since some isometries are not linear. Some of these nonlinear isometries maybe related to automorphisms of $GF(p^n)$; consequently, I believe there is an elegant structure underlining the isometries of graphs derived from finite fields. I also suspect the isometries of a semi-direct product involving graphs derived from finite fields is related to the similarities of graphs derived from finite fields, as theorem 2 demonstrated for GF(p). While understanding the isometries of semi-direct products involving finite fields is a big step, understanding the isometries of semi-direct products for any graphs may be too big.

Previous students have classified all two-point homogenous graphs up to 13 vertices. I am interested to know how much further the semi-direct product gets the classification and what the smallest number of vertices is where a graph is not part of any known family, nor derived from a group, nor any type of product of smaller graphs.

Bibliograph

[1] Hungerford, Thomas W. *Algebra*. Springer-Verlag: New York, 1974.