3. Prove the rest of Theorem 7.2.4.
4. a) In the models illustrated in Figs. 7.2 and 7.3 find the maximum number of points so that no three are on the same line.
   b) In an affine plane of order n, prove there can be at most $n+2$ points such that no three are on the same line.
   c) Repeat part (b) for a projective plane of order n.
5. Prove Theorem 7.2.5.
6. Prove Theorem 7.2.6. [*Hint:* Modify the proof of Theorem 7.2.3 and use duality.]
7. Prove Theorem 7.2.7. [*Hint:* Modify the proof of Theorem 7.2.4 and use duality.]
8. a) For which axiom of affine planes is its dual provable? Prove it.
   b) For the other axioms, show that their duals must be false.
   c) For which parts of theorems of affine planes are the duals provable? Prove them.
9. A *weak projective plane* satisfies Projective Axioms (i) and (iii) and the following replacements for Axiom (ii).
   ii′) There exist a point and a line not on that point.
   ii″) Every line has at least two points on it.
   a) Find a model of a weak projective plane of order 1.
   b) Prove that duality holds in this revised axiomatic system.
   c) Find a model of a weak projective plane with two lines having different numbers of points on them.
   d) Suppose that we replace Axiom (ii) for affine planes with the Axioms (ii′) and (ii″). Show that every such "weak affine plane" actually is an affine plane.

7.3 Design Theory

Design theory encompasses many notions besides affine and projective geometries. Design theory developed from mathematicians' natural inclination to generalize and explore and to meet the needs of applications, especially designs for statistical experiments. We concentrate on balanced incomplete block designs.

Some sampling trials require comparison of each of a number of varieties of something with all the other varieties available. Often one individual can't be expected to judge fairly among a large number of varieties. For example, taste comparisons by one person need to be done in a relatively short session. The organizer of the trials must guard against biases caused by which varieties are tasted together. Fisher and others developed designs in which each individual tests the same number of varieties (called the *size of the block*) and all pairs of varieties are compared the same number of times. Such designs are called balanced incomplete block designs (BIBD). *Balanced* refers to the uniformity of the arrangement, and *incomplete* refers to the fact that no block includes every variety. Fisher proved a number of results about BIBD. (See Anderson [1, Chapter 6].)

Remark Solutions to the n^2-officer problem are not BIBDs, although they are related, as in Example 2 of Section 7.2

Definition 7.3.1 A *balanced incomplete block design* (BIBD) is an arrangement of v varieties in b blocks (subsets), each of size k, so that each pair of varieties appear in λ blocks. We write (v, k, λ) to describe the numerical type of a BIBD.

Example 1 An affine plane of order n is a BIBD, with the lines as the blocks: $v = n^2$, $b = n^2 + n$, $k = n$, and $\lambda = 1$. Note that each point (variety) is on the same number, $r = n + 1$, of lines (blocks). ●

SIR RONALD A. FISHER

Sir Ronald A. Fisher (1890–1962) was one of the key people in establishing statistics as a mathematical discipline. Throughout his career he blended his interests in biology and statistics, building on his strong mathematical ability and insight. In 1911, as an undergraduate at Cambridge University, he gave a talk marking him as one of the first to see how to combine Darwin's theory of natural selection with Mendel's recently rediscovered ideas in genetics. He strongly advocated eugenics, the movement to improve people's genetic inheritance. (His involvement was innocent of and prior to the use of eugenics ideas for political and racist ends in various countries, most notoriously Nazi Germany.)

From an early age his eyesight was poor, which required active compensation. For example, in high school he studied spherical trigonometry entirely orally. His extraordinary visualization skills enabled him to perform all the three-dimensional thinking and computing without either a text or paper and pencil. His geometric intuition also characterized his statistical thinking. His first result, as well as many others, depended on considering a statistical sample of n points as a vector in n-dimensional Euclidean space. Many other statisticians, lacking Fisher's geometric abilities, found his reasoning difficult to follow and felt he depended too much on intuition. He made fundamental contributions in many areas of statistics, including tests of hypotheses and analysis of variance.

He combined his statistical, mathematical, and biological interests throughout his career. In 1919, he became the statistician at Rothamsted Experimental Station, where daily practical problems led him to a wide variety of theoretical discoveries in statistics. It was here that he realized the need both for randomized sampling and the careful design of experiments to study the interaction of variables. His biological experiments led to development of statistical design theory.

Example 2 A projective plane of order n is a BIBD, with the lines as the blocks: $v = n^2 + n + 1$, $b = n^2 + n + 1$, $k = n + 1$, and $\lambda = 1$. Again, each point is on the same number, $r = n + 1$, of lines. ●

Example 3 Given a BIBD $(v, k, 1)$ we can make a BIBD (v, k, λ) simply by repeating each block of the original BIBD λ times. However, experimenters often need to place a pair of varieties with different varieties in different blocks to gain additional information. The following list gives blocks for a BIBD (7, 3, 2) with no repetitions of blocks. Note that each line is a BIBD (7, 3, 1), so there are other ways to form a (7, 3, 2) BIBD by using repetition.

124	235	346	457	156	267	137
126	237	134	245	356	467	157 ●

Checking even the short list in Example 3 to verify that it is a (7, 3, 2) BIBD is tedious. It would be much worse with a larger design and harder still to find such a design by trial and error. Indeed, not every set of values (v, k, λ)—even those values that satisfy the combinatorial relations of Theorem 7.3.1—has a BIBD.