a) Show that v, the total number of girls, must satisfy $v = 12k + 4$.

b) Repeat Problem 4(b) for $v = 12k + 4$.

6. Generalize Problem 5, wherein there are n girls in each row.

7. A geometric figure **G** satisfies the following conditions.

i) **G** is made of points, some pairs of which are adjacent.

ii) Every point of **G** is adjacent to $k = 2$ other points.

iii) No three points of **G** are mutually adjacent.

iv) In **G** one point is adjacent to any two non-adjacent points.

a) Find a figure satisfying all four conditions. How many points are there?

b) Repeat part (a) for $k = 3$ in condition (ii). (This figure is called the Petersen graph.)

c) For a general k in part (ii), find a formula for the total number of points in a figure satisfying all four conditions.

d) For various values of k, find other figures that satisfy conditions (i), (ii), and (iv) but not (iii).

7.2 Affine and Projective Planes

7.2.1 Affine planes

Focusing on a few of the undefined terms and axioms of Euclidean or other geometries helps us understand these geometries better. This restriction also leads naturally to geometries having finitely many points. With affine planes we explore relations of points and lines, especially parallelism, but can't use them to consider betweenness, distance, and congruence. (Of course, mathematicians have explored these concepts in other geometric systems.) The axiomatic system of an *affine plane* has the undefined terms *point*, *line*, and *on* and the following axioms.

Axioms 7.2.1

i) Every two distinct points have exactly one line on them both.

ii) There are at least four points with no three on the same line.

iii) For every line and point not on that line, there is a unique line on that point that has no point in common with the given line.

Exercise 1 Verify that the Euclidean plane is an affine plane.

Definition 7.2.1 Two lines, k and l, are *parallel*, written $k \| l$, iff either $k = l$ or no point is on both k and l.

Definition 7.2.2 An affine plane with n points on each line is of *order* n.

Example 1 Figure 7.2 depicts a model of an affine plane of order 2 and Fig. 7.3 depicts a model of order 3. In each figure, lines parallel to one another are drawn with the same kind of line for clarity. We can use Fig. 7.3 to obtain solutions to the nine-schoolgirl problem in Problem 3 of Section 7.1. For example, for the first day the rows could be the horizontal lines, for the second day the rows could be the vertical lines, and so on. ●

After proving some initial, elementary properties of all affine planes in Theorems 7.2.1 and 7.2.2, in Theorem 7.2.3 we show that all finite affine planes have an order, as stated in Definition 7.2.2. In Theorem 7.2.4 we use a combinatorial argument

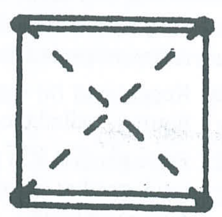

Figure 7.2 An affine plane of order 2.

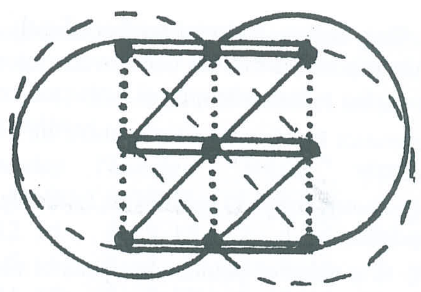

Figure 7.3 An affine plane of order 3.

to reveal necessary relationships between the numbers of points and lines. However, such counting arguments can't guarantee the existence of affine planes.

Theorem 7.2.1 In an affine plane

- **i)** two distinct lines have at most one point in common,
- **ii)** for every point there is a line not on that point,
- **iii)** for every line there is a point not on that line,
- **iv)** every line has at least two points on it, and
- **v)** every point is on at least three lines.

Proof. For parts (iii) and (iv), let k be any line. Let A, A_1, A_2, and A_3 be the four points guaranteed by Axiom (ii). At least one of these four points, say, A, is not on k, showing part (iii). Consider the lines k_i, which are on A and A_i. Axiom (ii) ensures that these lines are distinct. At most, one of them is parallel to k, by Axiom (iii) and hence the other two intersect k. These two lines already have A in common, so they must intersect k in different points by part (i). See Problem 1 for the remaining parts. ■

Theorem 7.2.2 Parallelism is an equivalence relation for lines in an affine plane. That is, for all lines k, l, and m, three properties hold: reflexive, $k \| k$; symmetric, if $k \| l$, then $l \| k$; and transitive, if $k \| l$ and $l \| m$, then $k \| m$.

Proof. See Problem 2. ■

Theorem 7.2.3 If some line of an affine plane has n points on it, then each line has n points on it and each point has $n + 1$ lines on it.

Exercise 2 Draw diagrams to illustrate the proof of Theorem 7.2.3.

Proof. Let k be a line with n points on it, say, $P_1, \ldots, P_n$, where $n \geq 2$. First, let l be any line not parallel to k. By Axiom (ii), there is a point Q on neither l nor k. By Axiom (i), Q has n lines on it that intersect k plus a parallel to k, by Axiom (iii), for a total of $n + 1$ lines. Theorem 7.2.1 guarantees all to be distinct lines. In turn, Axiom (iii) forces all but one line to be on l, so l has n points on it. Now suppose that a line m is parallel to k. Use the axioms to find a line j not parallel to k. We can use the transitive property of

parallelism to show that m is not parallel to j. (See Problem 2(b).) Now, from the first part of this proof, we know that j must have the same number of points as k and as m. Hence m has the same number of points as k. The preceding argument for the point Q shows that any point not on line k has $n + 1$ lines on it. For a point on line k, the same reasoning holds with a line not on that point. ■

Exercise 3 What happens in Theorem 7.2.3 for a line with infinitely many points on it?

Theorem 7.2.4 In an affine plane if some line has n points on it, then there are n^2 points and $n^2 + n$ lines, and each line has n lines parallel to it, including itself.

Proof. Let P be any point. Then there are $n + 1$ lines on P and each of them has $n - 1$ points on it other than P by Theorem 7.2.3. Thus there are $(n + 1)(n - 1)$ points besides P, giving a total of $(n + 1)(n - 1) + 1 = n^2$ points. See Problem 3 for the rest of this proof. ■

Example 2 There is no affine plane of order six.

Solution. We show that an affine plane with six points on a line would provide a solution to Euler's 36-officer problem, which we assume to be impossible. So, for a contradiction, suppose that there were an affine plane of order 6. We choose any two nonparallel lines to determine the rows and columns of the officers and fix any one point P (Fig. 7.4). Pick a line k through P other than a row or column. Put all the officers from the first regiment on k and from each other regiment on one of the parallels of k. Then no two officers from the same regiment would be in the same row or column by Theorem 7.2.1. Next, pick a line m through P other than k or a row or a column. Put the officers of each rank on one of the parallels of m. Again, we would have one officer from each rank in each row and column. This outcome contradicts the fact that there is no solution to Euler's problem. Hence there can be no affine plane of order 6. For a direct proof, see Anderson [1, 91]. ●

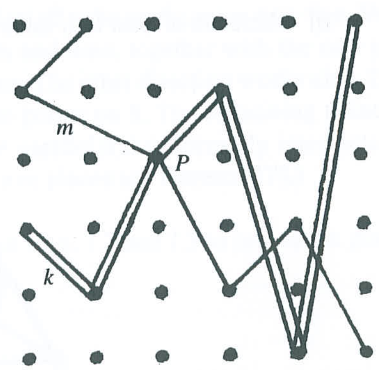

Figure 7.4 Using a hypothetical affine plane of order 6 to solve the 36-officer problem.

The idea behind Example 2 enabled Fisher to turn affine planes into a solution of the statistical design problems discussed in Section 7.1. But for which orders are there actually affine planes? By 1896, geometers had shown algebraically that affine planes (and projective planes) of order n exist if $n = p^k$, where p is any prime number. In Section 7.4 we develop analytic geometries with this approach. No other orders of n are known to give planes, and many geometers believe that no other finite orders are possible. Theoretical results have eliminated infinitely many other values of n. An extensive computer search in 1989 revealed that there is no affine or projective plane of order 10, the smallest order not previously decided theoretically. Proving which orders of n give planes is one of the key open problems in finite geometry. (See Mullen [8].)

7.2.2 Projective planes

Axiom (iii) for affine planes characterizes parallelism. Projective geometry focuses on a contrasting idea from perspective drawings in art, whereby parallel lines appear to meet at a point on the horizon. Axiom (iii) characterizes projective planes and is the only change from the axioms of affine planes. (The axioms here are the first three axioms of the real projective plane in Chapter 6.) The axiomatic system for projective planes has the undefined terms *point*, *line*, and *on* and the following axioms.

Axioms 7.2.2

- **i)** Every two distinct points have exactly one line on them both.
- **ii)** There are at least four points with no three on the same line.
- **iii)** Every two lines have at least one point on them both.

Exercise 4 Explain why spherical geometry isn't a model of these axioms but single elliptic geometry is.

Example 3 Figure 7.5 gives a model of a projective plane with three points on each line. ●

Theorem 7.2.5 In a projective plane

- **i)** two distinct lines have one point on them both,
- **ii)** there are at least four lines with no three on the same point,

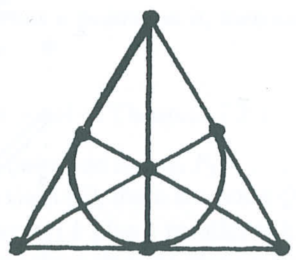

Figure 7.5 A projective plane of order 2. (The curve is the seventh line.)

- **iii)** every line is on at least three points and
- **iv)** every point is on at least three lines.

Proof. See Problem 5. ∎

Projective planes have an important and mathematically aesthetic property called *duality*. Every theorem about points and lines remains a theorem when we systematically switch the words *point* and *line*. The original theorem and the switched theorem are called *duals* of each other. The first two parts of Theorem 7.2.5 contain the duals of Axioms (i) and (ii). [The dual of Axiom (iii) is weaker than that of Axiom (i), so this dual holds.] Because the duals of the axioms are all theorems, whenever we prove a theorem in projective geometry we immediately obtain its dual by switching the words *point* and *line* in the proof.

Theorem 7.2.6 If one line of a projective plane has $n + 1$ points on it, then all lines have $n + 1$ points on them and all points have $n + 1$ lines on them.

Proof. See Problem 6. ∎

Theorem 7.2.7 If one line of a projective plane has $n + 1$ points on it, there are $n^2 + n + 1$ points and $n^2 + n + 1$ lines.

Proof. See Problem 7. ∎

Definition 7.2.3 A projective plane of *order n* has $n + 1$ points on a line.

A natural connection exists between affine and projective planes of the same order, explaining Definition 7.2.3. Theorems 7.2.4 and 7.2.7 show that a projective plane of order n has one more line and $n + 1$ more points than an affine plane of the same order. The following construction shows that we can convert an affine plane into a projective plane of the same order by adding one line and its points. This result corresponds to the addition of a horizon line in perspective drawing. For an affine plane of order n, we collect the lines parallel to each other in a class. For each of the $n + 1$ classes of parallel lines, we add a new point that we define to be on each of those parallel lines. We also define these $n + 1$ new points all to be on the same new line. We leave it as an exercise to verify that the affine points and lines, together with the new points and line, satisfy the axioms of a projective plane. The other direction works also. Take a projective plane and delete any one line and the points on it. The remaining points and lines form an affine plane, where lines become parallel if they formerly intersected on the deleted line. (For more on affine and projective planes see Karteszi [7].)

Exercise 5 Convert the affine planes in Figs. 7.2 and 7.3 to projective planes using the construction given above.

PROBLEMS FOR SECTION 7.2

1. Prove the rest of Theorem 7.2.1.
2. a) Prove Theorem 7.2.2.
   b) If a line in an affine plane intersects one of two parallel lines, prove that it intersects the other.

3. Prove the rest of Theorem 7.2.4.
4. a) In the models illustrated in Figs. 7.2 and 7.3 find the maximum number of points so that no three are on the same line.
   b) In an affine plane of order n, prove there can be at most $n + 2$ points such that no three are on the same line.
   c) Repeat part (b) for a projective plane of order n.
5. Prove Theorem 7.2.5.
6. Prove Theorem 7.2.6. [*Hint:* Modify the proof of Theorem 7.2.3 and use duality.]
7. Prove Theorem 7.2.7. [*Hint:* Modify the proof of Theorem 7.2.4 and use duality.]
8. a) For which axiom of affine planes is its dual provable? Prove it.
   b) For the other axioms, show that their duals must be false.
   c) For which parts of theorems of affine planes are the duals provable? Prove them.
9. A *weak projective plane* satisfies Projective Axioms (i) and (iii) and the following replacements for Axiom (ii).
   ii′) There exist a point and a line not on that point.
   ii″) Every line has at least two points on it.
   a) Find a model of a weak projective plane of order 1.
   b) Prove that duality holds in this revised axiomatic system.
   c) Find a model of a weak projective plane with two lines having different numbers of points on them.
   d) Suppose that we replace Axiom (ii) for affine planes with the Axioms (ii′) and (ii″). Show that every such "weak affine plane" actually is an affine plane.