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Finite Geometries

Experimenters testing varieties of seed, fertilizers, or pesticides need to guard against other factors accidentally favoring one variety over another. For example, the drainage or quality of the soil will vary from one part of a field to another. Thus differences in the success of different varieties might be due to extraneous factors rather than the varieties themselves. Finite geometries provide a way to minimize such potential bias in experiments in areas ranging from agriculture to taste testing. Mathematicians have also used finite geometries to develop error correcting codes.

> The moving power of mathematical invention is not reasoning but imagination. —*Augustus DeMorgan*

7.1 Overview and History

Finite geometries as a subject arose from the investigations of geometric axioms at the end of the nineteenth century. The advent of hyperbolic and other geometries prompted a renewed interest in axioms. In particular, geometers sought models satisfying certain axioms but not others. Often these models had finitely many points, whence the term finite geometries. The first finite geometry was a three-dimensional example with 15 points developed in 1892 by Gino Fano. Geometers soon realized that finite geometries and their axiom systems were interesting for their own sake. Geometers incorporated into them various intriguing designs, results, and problems that had preceded this explicit development of finite geometry.

As so often happens in mathematics, significant applications arose from the blend of interesting problems and mathematical structure. In turn these applications have enriched mathematics by suggesting new areas to explore. Statistical design theory drew on finite geometries and expanded the types of geometric structures and theorems. More recently computers have greatly influenced the study of finite geometries, both by the ability to consider more complicated examples and to meet the need for sophisticated designs. Examples 1 and 2 illustrate that brute computing force may solve particular problems but that more general problems require mathematical insight.

Example 1 Euler posed the "36-officer problem" in 1782. He imagined six military regiments, each with the same six ranks and supposed one officer of each rank from each regiment. Euler sought a square array of these 36 officers so that no two officers of the same rank or from the same regiment were in the same row or column. He was convinced that this problem had no solution but was unable to give a proof. (The impossibility was finally proved in 1900 by an exhaustive search.) He was able to construct a solution to the corresponding problem for $5^2 = 25$ officers, among others (Fig. 7.1). He also conjectured that similar problems with n^2 officers, where $n = 4k + 2$, would be impossible. Euler's intuition was uncharacteristically flawed. In 1960 after extensive research, mathematicians showed that they could solve the n^2 officers problem for every value of n except 2 and 6. ●

$A1$	$B2$	$C3$	$D4$	$E5$
$D3$	$E4$	$A5$	$B1$	$C2$
$B5$	$C1$	$D2$	$E3$	$A4$
$E2$	$A3$	$B4$	$C5$	$D1$
$C4$	$D5$	$E1$	$A2$	$B3$

Figure 7.1 A solution to the 25-officer problem. The five ranks are A, B, C, D, and E, and the five regiments are 1, 2, 3, 4, and 5.

LEONHARD EULER

The Swiss mathematician Leonhard Euler (1707–1783) was the most prolific mathematician of all time—it took the St. Petersburg Academy 47 years to finish publishing the more than two hundred papers left when he died. Euler (pronounced "oiler") finished his university degree in theology at age 15. Meanwhile he had started studying mathematics with Johann Bernoulli and published his first paper at age 18. He followed Bernoulli to the St. Petersburg Academy in Russia and stayed there until he was 34. Euler then went to Berlin for 25 years at the request of Frederick the Great, [illegible] mathematician should be at the court of the greatest king." Euler returned to Russia in 1766 blind in one eye and soon became completely blind. However, his production of mathematics didn't diminish. He merely dictated his papers and used his phenomenal memory to accomplish even the most complicated computations in his head. (Euler created the 36-officer problem, with its strong visual appeal, when he was blind.) Euler knew and made major contributions to all of mathematics and many of its applications in physics, although we discuss only some of his geometric work.

Euler, like mathematicians of his time, focused on particular problems which led to deeper, general insights. Euler's investigation of the Königsberg Bridge problem in 1735 marks the beginning of graph theory, an area of mathematics closely tied to combinatorics. Euler's formula (and its generalizations), discussed in Section 1.6, is a key theorem in graph theory and topology. Euler gave his argument for this formula in 1751. Euler's many textbooks established the importance of functions in analytic geometry and calculus. In his 1748 textbook he first introduced parametric equations (see Section 2.3). The problem of representing the curved surface of the earth on flat maps led to a variety of Euler's investigations and differential geometry.

Euler's fame as a mathematician provided one reason to tackle the general problem in Example 1, but a modern application provided another impetus. In the 1920s Sir Ronald A. Fisher developed statistical design theory. He and others wanted to test, for example, how the interactions of varieties of fertilizers and pesticides affected the growth of a type of plant. They needed to test each variety of fertilizer with each variety of pesticide. For n varieties of each, there are n^2 combinations. Previously, others had arranged different varieties of fertilizer in the rows and different varieties of pesticide in the columns. However, fields in nature, unlike mathematics, aren't uniform. Hence some rows or columns may provide better drainage, soil nutrients, or other growing conditions than others. To control for such "confounding variables," Fisher realized that he needed a design wherein each type of fertilizer and each type of pesticide appeared once in each row and column. Thus Euler's 36-officer problem became the prototype for Fisher's design problems. Fortunately, geometers had already solved many of these types of problems by using affine planes, which we develop in Sections 7.2 and 7.4. Fisher used these geometric solutions, starting a vital interaction between finite geometries and design theory.

Example 2 In 1850, the Rev. Thomas Kirkman posed and solved the "15-schoolgirl problem." He supposed that the girls took walks every day in an artificially regimented style of five rows of three each. He asked for daily arrangements of them so that any two girls walked in the same row just once a week. Computers have listed all 845 essentially different ways to solve this problem, but there is no easy way to generate such a solution. One such solution, from Berman and Fryer [3], follows.

Sunday	*Monday*	*Tuesday*	*Wednesday*	*Thursday*	*Friday*	*Saturday*
1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
4 8 12	2 8 10	2 9 11	2 12 14	2 13 15	2 4 6	2 5 7
5 10 15	3 13 14	3 12 15	3 5 6	3 4 7	3 9 10	3 8 11
6 11 13	6 9 15	4 10 14	4 11 15	5 9 12	5 11 14	4 9 13
7 9 14	7 11 12	5 8 13	7 10 13	6 8 14	7 8 15	6 10 12

Which of the numerical conditions in this problem imply the others? The total number of girls (15) and the size of each row (3) clearly determine the number of rows (5). We use the condition that each pair of girls occurs just once to determine the number of days possible. For any girl, the 14 other girls must be put with this girl in pairs so that there can only be seven nonrepeating days. This reasoning is a simple example of combinatorics, a branch of mathematics that provides insight into numerical relationships. ●

Finite geometry is currently an active area of research benefiting from the cross-fertilization of geometry, algebra, and combinatorics. Transformational geometry and group theory gave powerful insights about finite geometries in the twentieth century just as they have since the nineteenth century for traditional geometries. Combinatorics provides essential insights into finite and other areas of geometry. For example, the consequences of Euler's formula (see Chapter 1) were identified through combinatorial reasoning. Error-correcting codes from coding theory, which we discuss in Section 7.3, are used in electronic data transmission and benefit from the interaction of these areas of mathematics.

PROBLEMS FOR SECTION 7.1

1. **a)** Find a solution to the "9-officer" problem.
   b) Find a solution to the "16-officer" problem.
   c) Explain why there can be no solution to the "4-officer" problem.
2. **a)** Describe how the *A*s, *B*s, and other letters in Fig. 7.1 relate to each other. Similarly describe the placement of the numbers.
   b) Find a different solution to the "25-officer" problem and describe any patterns you find in the placement of the letters and numbers.
   c) Does Fig. 7.1 avoid the risk of confounding variables that Fisher needed for his statistical work? Repeat for your solution in part (b). Discuss your answers.
3. Modify Example 2 so there are nine girls. Find daily arrangements of nine girls in rows of three so that each pair of girls appears in the same row just once. How many days can they go walking before they repeat partners?
4. Generalize Example 2 to a v-schoolgirl problem, wherein there are still three girls in each row and every two girls are in the same row just once.
   a) Show that v must be an odd multiple of 3, say, $v = 6n + 3$.
   b) For $v = 6n + 3$, find the number of days that these girls can go for walks without any girl walking with another girl twice. (Combinatorics won't guarantee such arrangements of these $6n + 3$ girls.)
5. Generalize Problem 4 so that there are four girls in each row but that two girls are still in the same row just once.
   a) Show that v, the total number of girls, must satisfy $v = 12k + 4$.
   b) Repeat Problem 4(b) for $v = 12k + 4$.
6. Generalize Problem 5, wherein there are n girls in each row.
7. A geometric figure **G** satisfies the following conditions.
   i) **G** is made of points, some pairs of which are adjacent.
   ii) Every point of **G** is adjacent to $k = 2$ other points.
   iii) No three points of **G** are mutually adjacent.

   iv) In **G** one point is adjacent to any two non-adjacent points.
   a) Find a figure satisfying all four conditions. How many points are there?
   b) Repeat part (a) for $k = 3$ in condition (ii). (This figure is called the Petersen graph.)
   c) For a general k in part (ii), find a formula for the total number of points in a figure satisfying all four conditions.
   d) For various values of k, find other figures that satisfy conditions (i), (ii), and (iv) but not (iii).