6.6 PROJECTIVE SPACE

Homogeneous coordinates and collineations can be readily extended to higher dimensional projective spaces, which are significant models that are used for many purposes. For example, three-dimensional projective space provides perspective views for computer-aided design (CAD). And the Lorentz transformations in the special theory of relativity can be seen as isometries in a subgeometry of four-dimensional projective space related to hyperbolic geometry.

Interpretation By a *point* in n-dimensional projective space $\mathbf{P}^n$ we mean a one-dimensional subspace of the $n+1$ dimensional vector space $\mathbf{R}^{n+1}$. By *line* we mean a two-dimensional subspace, by *plane* we mean a three-dimensional subspace, and so on. A point (line, and so on) is *on* a line (plane, and so on) iff one is a subset of the other. By a *collineation* in $\mathbf{P}^n$ we mean an invertible $(n+1)\times(n+1)$ matrix. Any nonzero scalar multiple of a matrix represents the same collineation.

Example 1 Two points in $\mathbf{P}^3$ (or any $\mathbf{P}^n$) have a unique line on them. In other words, two one-dimensional subspaces are spanned by a unique two-dimensional subspace. However in $\mathbf{P}^3$ two lines can fail to intersect, which corresponds to skew lines in Euclidean space. For example, the lines (two-dimensional subspaces) $\{(x, y, 0, 0) : x, y \in \mathbf{R}\}$ and $\{(0, 0, z, t) : z, t \in \mathbf{R}\}$ intersect only at the origin $(0, 0, 0, 0)$, the zero-dimensional subspace, which isn't a projective point. Two distinct planes (three-dimensional subspaces) in $\mathbf{P}^3$ must intersect in a line (a two-dimensional subspace). Each plane has three basis vectors, giving six possible vectors. The whole space needs only four basis vectors, so there must be an overlap of at least two. Because the planes are distinct, the overlap is exactly two—a line. In $\mathbf{P}^3$, point and planes are duals and a line is "self-dual." Thus the dual in $\mathbf{P}^3$ of "Two points determine a line" is "Two planes determine a line." In general, the dual of an i-dimensional subspace in $\mathbf{P}^n$ is an $(n+1-i)$-dimensional subspace. ●

6.6.1 Perspective and computer-aided design

A general collineation in $\mathbf{P}^3$ (a 4×4 matrix) can be broken into component parts, most of which appear in affine transformations. Recall that the rightmost column of an affine matrix describes how the origin moves and corresponds to a translation. The upper left 3×3 submatrix determines the type of affine transformation: rotation, reflection, shear, dilation, and so on. The bottom row of an affine matrix is $[\,0\;\;0\;\;0\;\;1\,]$. The bottom row of a collineation provides flexibility lacking in affine transformations. In a CAD system the change of the lower right entry from a 1 to another value magnifies or shrinks the entire picture by the same factor—a much faster computer alteration than changing all the upper entries by the reciprocal factor. The first three entries of the bottom row determine perspective views in each dimension. The matrix $\begin{bmatrix} a & a & a & t_x \\ a & a & a & t_y \\ a & a & a & t_z \\ p_x & p_y & p_z & s \end{bmatrix}$ summarizes this discussion, where a stands for affine, p for perspective, s for scaling, and t for translation. (See Penna and Patterson [5] for more information on projective geometry and computer graphics.)

Example 2 We illustrate the effects of the perspective entries by using several related matrices to project a cube. (For convenience we ignore the translation and scaling entries.) By Theorem 4.5.4, we need only consider what the transformations do to the 4×8 matrix $V = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ containing the eight vertices of the cube.

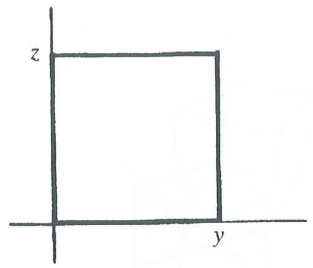

Figure 6.15 Image of a cube under PV.

The matrix $P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ represents a projection parallel to the x-axis; it isn't a collineation because it collapses three-dimensional objects to two dimensions. On a computer screen, in nonhomogeneous coordinates the eight vertices in PV, together with their edges, would appear as a square (Fig. 6.15). To see the three-dimensional form of a cube we need to have a different viewing angle. Hence we rotate the cube $-30°$ around the z-axis and then $-30°$ around the (original) y-axis, using the matrix $R = \begin{bmatrix} \cos -30 & 0 & -\sin -30 & 0 \\ 0 & 1 & 0 & 0 \\ \sin -30 & 0 & \cos -30 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos -30 & -\sin -30 & 0 & 0 \\ \sin -30 & \cos -30 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/4 & \sqrt{3}/4 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & 0 \\ -\sqrt{3}/4 & -1/4 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then PRV gives the familiar, nonperspective view of the cube (Fig. 6.16). Note that all the faces in Fig. 6.16 are parallelograms. Let $P_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/5 & 0 & 0 & 1 \end{bmatrix}$, $P_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/5 & -1/5 & 0 & 1 \end{bmatrix}$, and $P_{xyz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/5 & -1/5 & -1/5 & 1 \end{bmatrix}$. Then PRP_xV gives a one-point perspective view (Fig. 6.17), $PRP_{xy}V$ gives a two-point perspective view (Fig. 6.18), and $PRP_{xyz}V$ gives a three-point perspective view (Fig. 6.19).

In Fig. 6.17, note that the sides of the cube parallel to the x-axis all meet at a *vanishing point* denoted V_x. The other sides remain parallel. We can compute the coordinates of this vanishing point by finding the projection of the ideal point along the x-axis, $(1, 0, 0, 0)$: $PRP_x(1, 0, 0, 0) = (0, -\frac{1}{2}, -\frac{\sqrt{3}}{4}, -\frac{1}{5}) = (0, 2.50, 2.17, 1)$, or, in the nonhomogeneous coordinates of the figure, $(2.50, 2.17)$. Figure 6.18 includes the vanishing points for both the x- and y-axes, V_x and V_y. Actually, P_{xy} automatically gives vanishing points for all directions in the xy-plane, which appear on the "horizon line"

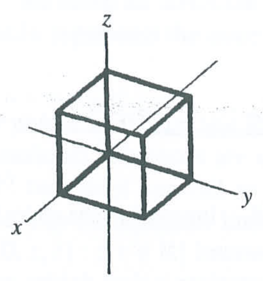

Figure 6.16 Image of a cube under *PRV*.

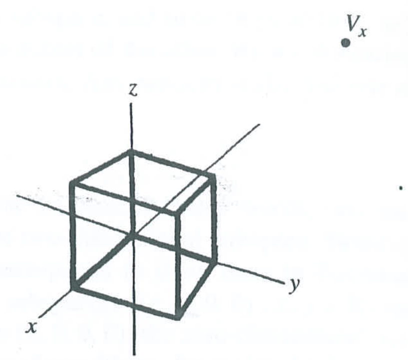

Figure 6.17 Image of a cube under PRP_xV.

$\overleftrightarrow{V_xV_y}$ shown in Fig. 6.18. For example, the diagonals connecting (0, 0, 1, 1) to (1, 1, 1, 1) and (0, 0, 0, 1) to (1, 1, 0, 1) meet at the vanishing point (−0.92, 1.71), the projection of the ideal point (1, 1, 0, 0). In three-point perspective (Fig. 6.19) every direction has a vanishing point. ●

Projective geometry, like single elliptic geometry, is not oriented. (See Section 3.5.) In effect, there is no consistent way to define clockwise rotations around a point or direction on a line. One consequence of this nonorientability is that collineations can alter the appearance of a figure beyond what an artist or computer operator needs. For example, projectivities (and so collineations) can map any three collinear points to any three collinear points. Therefore these transformations can alter the betweenness relations of points on a line. However, no perspective view of a real object will ever turn it inside out. Stolfi [7] has created an oriented projective geometry for CAD by not allowing negative scalars. We place a tilde ($\tilde{\ }$) over a point to indicate an oriented point. An *oriented point* is still a vector in $\mathbf{R}^4$, but two such vectors $\vec{v}$ and $\vec{w}$ represent the same point only if there is a positive real number k such that $\vec{w} = k \cdot \vec{v}$. In effect each

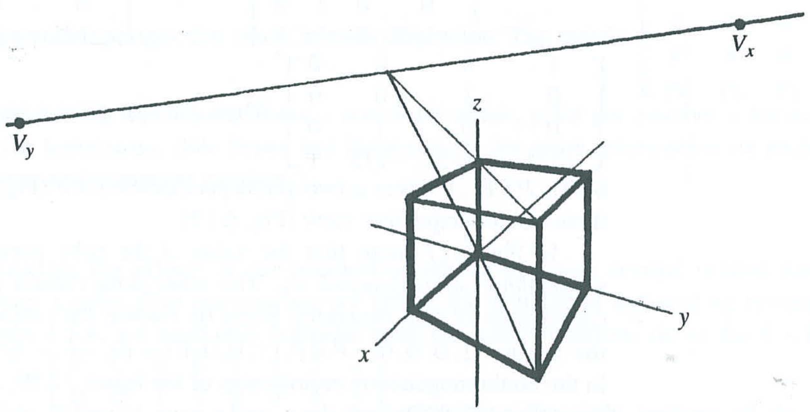

Figure 6.18 Image of a cube under $PRP_{xy}V$.

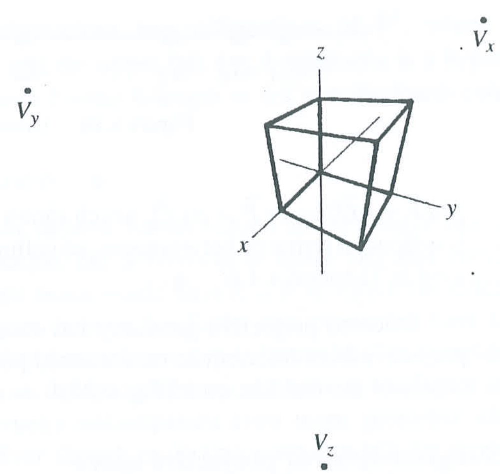

Figure 6.19 Image of a cube under $PRP_{xyz}V$.

projective point P is split into two opposite oriented points $\tilde{P}$ and $-\tilde{P}$. This corresponds exactly to the relationship between spherical and single elliptic geometry: Two opposite points in spherical geometry represent the same point in single elliptic geometry. All invertible matrices are still collineations in oriented projective geometry, although their effect changes and two matrices represent the same collineation when they differ by a positive scalar.

Definition 6.6.1 $\tilde{R}$ is *between* $\tilde{P}$ and $\tilde{Q}$ iff there are positive real numbers a and b such that $\tilde{R} = a\tilde{P} + b\tilde{Q}$. The line segment $\overline{\tilde{P}\tilde{Q}}$ is the set $\{\tilde{R} : \tilde{R}$ is between $\tilde{P}$ and $\tilde{Q}$, $\tilde{R} = \tilde{P}$ or $\tilde{R} = \tilde{Q}\}$. A set S of oriented points is *convex* iff, for all $\tilde{P}$ and $\tilde{Q}$ in S, $\overline{\tilde{P}\tilde{Q}}$ is a subset of S.

Example 3 The oriented points between $\widetilde{(0, 0, 1)}$ and $\widetilde{(1, 0, 1)}$ are $\widetilde{(x, 0, 1)}$, where $0 < x < 1$.

Proof. Let $\widetilde{(u, v, w)}$ be between $\widetilde{(0, 0, 1)}$ and $\widetilde{(1, 0, 1)}$. Then there are positive reals a and b such that $\widetilde{(u, v, w)} = a\widetilde{(0, 0, 1)} + b\widetilde{(1, 0, 1)} = \widetilde{(b, 0, a + b)} = \widetilde{(\frac{b}{a+b}, 0, 1)}$. Note that $x = b/(a + b)$ must be between 0 and 1 because both a and b are positive. WLOG we can pick a and b so that $a + b = 1$. Then (a, b) are the barycentric coordinates of $\widetilde{(x, 0, 1)}$ in terms of $\widetilde{(0, 0, 1)}$ and $\widetilde{(1, 0, 1)}$. (See Section 2.3 for a discussion of barycentric coordinates.) ●

Remark If we used these same definitions with regular projective points P and Q and any nonzero scalars, the entire line $\overleftrightarrow{PQ}$ would be "between" P and Q and so would be a "line segment." Theorem 6.6.1 would still be provable (changing the word *positive* to *nonzero*), but that would not be particularly helpful for the only convex sets would be a single point, a (projective) line, a (projective) plane, and the like.

Theorem 6.6.1 Collineations in oriented projective geometry preserve betweenness and convexity.

Proof. Let γ be a collineation and $\tilde{P}$ and $\tilde{Q}$ be any two points. Then $\tilde{R} = a\tilde{P} + b\tilde{Q}$, for a and b positive, is between $\tilde{P}$ and $\tilde{Q}$. Because γ is a linear transformation, $\gamma\tilde{R} =$

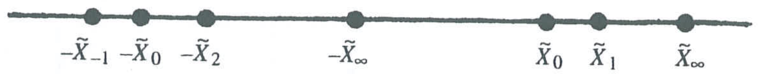

Figure 6.20 Oriented points on a line.

$\gamma(a\widetilde{P} + b\widetilde{Q}) = a\gamma\widetilde{P} + b\gamma\widetilde{Q}$, which shows that $\gamma\widetilde{R}$ is between $\gamma\widetilde{P}$ and $\gamma\widetilde{Q}$. Convexity is defined in terms of betweenness, so collineations preserve convexity by the argument used in Theorem 4.4.6. ∎

Oriented projective geometry has some peculiarities. For example, two oriented projective lines intersect in two oriented points. Also an oriented line in effect has two copies of the real line on it (Fig. 6.20).

6.6.2 Subgeometries of projective space

Euclidean, hyperbolic, single elliptic, and other geometries of n dimensions are subgeometries of the projective geometry of the same number of dimensions. The transformation groups are entirely analogous to the corresponding groups in two dimensions. We briefly consider three-dimensional hyperbolic space to lead into Minkowski geometry, used in the special theory of relativity.

The points of three-dimensional hyperbolic space are the points in the interior of the unit sphere $x^2 + y^2 + z^2 - t^2 = 0$, which we take as the absolute quadratic surface. Hyperbolic isometries leave this surface stable.

Exercise 1 Extend the definitions from Section 6.5 of h-inner product, h-orthogonal, and h-length to vectors in $\mathbf{R}^4$.

Theorem 6.6.2 A 4×4 nonsingular matrix represents a hyperbolic isometry iff its columns are h-orthogonal, the first three have the same h-length, and the last column has the opposite h-length.

Proof. See Problem 5. ∎

In Section 5.5 we discussed the Lorentz transformations, which preserve $\Delta x_A^2 + \Delta y_A^2 + \Delta z_A^2 - \Delta t_A^2$, a value closely related to the equation of the quadratic surface for hyperbolic space. Recall that the equation $\Delta x_A^2 + \Delta y_A^2 + \Delta z_A^2 - \Delta t_A^2 = \Delta x_B^2 + \Delta y_B^2 + \Delta z_B^2 - \Delta t_B^2$ expressed the invariance of the distance and time measurements between two events from two frames of reference moving at a constant velocity relative to each other. Transformations preserving all values $k = \Delta x_A^2 + \Delta y_A^2 + \Delta z_A^2 - \Delta t_A^2$ clearly leave $x^2 + y^2 + z^2 - t^2 = 0$ stable and so are related to hyperbolic isometries. Actually, the four-dimensional geometry for relativity, called Minkowski geometry, needs all points (x, y, z, t). As with affine geometry, an extra coordinate is needed to permit movement of the origin. Hence the points are written $(x, y, z, t, 1)$. Thus Minkowski geometry is a subgeometry of $\mathbf{P}^4$, rather than $\mathbf{P}^3$. A Lorentz transformation is therefore a collineation of $\mathbf{P}^4$, but its upper left 4×4 submatrix is a hyperbolic isometry. As in affine transformations, the bottom row is $[\,0\ \ 0\ \ 0\ \ 0\ \ 1\,]$ and the right column represents translations of the origin.

Theorem 6.6.3 The Lorentz transformations are collineations of $\mathbf{P}^4$, where the bottom row is $[\,0\ \ 0\ \ 0\ \ 0\ \ 1\,]$ and the upper left 4×4 submatrix is a hyperbolic isometry, with the first three columns having h-length $= \pm 1$ and the fourth column having h-length $= \mp 1$.

Proof. See Problem 6. ∎

In Section 5.5 we showed that the value k above could be zero if the two events were "lightlike." For example, the differences in space and time coordinates of two points on the path of a light beam would have $k = 0$. In effect, the h-length of the difference of two lightlike events is zero. Similarly, two events can have k positive or negative depending on whether their relationship is "spacelike" or "timelike," respectively. (For further information on the special theory of relativity see Taylor and Wheeler [8].)

Projective geometry encompasses even more geometric ideas than Cayley and Klein envisioned. Even though projective geometry falls far short of "all geometry," as Cayley exclaimed in the quote opening this chapter, it has proven its worth in classical geometry, CAD systems, and many other applications.

PROBLEMS FOR SECTION 6.6

1. Find and prove a condition similar to that in Example 4 of Section 6.3, describing when four points of $\mathbf{P}^3$, projective space, are in the same plane.
2. Decide for which values of n the sets in parts (a)–(c) must intersect in $\mathbf{P}^n$. Justify your answers.
   a) A line and a plane
   b) Two planes
   c) A k-dimensional subspace and a j-dimensional subspace
   d) What happens to the intersection in part (c) as n decreases?
3. Draw a three-point perspective image of the cube in Example 2, using -0.25 for the first three entries in the bottom row. Find the vanishing points V_x, V_y, and V_z. What effect does changing the entries from -0.2 to -0.25 have?
4. Draw a three-point perspective image of the cube in Example 2, using $+0.2$ for the first three entries in the bottom row. Find the vanishing points V_x, V_y, and V_z. What effect does changing these entries from -0.2 to $+0.2$ have?
5. Prove Theorem 6.6.2. [*Hint:* See Example 3 of Section 6.5.]
6. Prove Theorem 6.6.3.
7. Prove that the intersection of any collection of convex sets in oriented projective geometry is again a convex set.
8. **a)** Graph the circle $x^2 + y^2 = 4$ and the hyperbola $x^2 - y^2 = 1$ and shade in the interiors of the circle $x^2 + y^2 < 4$ and of the hyperbola $x^2 - y^2 > 1$. Note that the intersection of their interiors has two separate regions, which therefore can't form a convex set.
   b) Explain why the oriented points $\widetilde{(x, y, z)}$, with $x^2 + y^2 < 4z^2$ and $z > 0$, are interior to one oriented representation of the circle in part (a) and why these oriented points form a convex set.
   c) Verify that the collineation $\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}$ takes the circle of part (a) to the hyperbola of part (a). Verify that points interior to the circle are taken to points interior to the hyperbola.
   d) Explain how to resolve the following seeming contradiction between Theorem 6.6.1, Problem 7, and the preceding parts of this problem. The oriented interior of the circle is convex, the circle is mapped to the hyperbola, and Theorem 6.6.1 implies that the hyperbola's oriented interior is convex. Problem 7 says that the intersection of two such convex sets must be convex, yet the intersection in part (a) clearly is not convex.

PROJECTS FOR CHAPTER 6

1. Investigate how artists make perspective drawings. (See Powell [6].)
2. Use parts (a) and (b) and Fig. 6.21 to explain why Desargues's theorem holds in three-dimensional Euclidean space. Desargues's theorem: If two triangles are perspective from a point, then they are perspective from a line.
   a) First let triangles $\triangle ABC$ and $\triangle DEF$ be in two nonparallel planes and perspective from P. Explain why the plane through P, A, and B must include D, E, and R. Explain why the two triangles must be perspective from the line on the intersection of the planes containing the triangles.
   b) Explain how to use part (a) twice to prove Desargues's theorem if the two triangles are in parallel planes or the same plane.
   c) What adjustments would be needed in this argument for three-dimensional projective space?
   d) State the converse of Desargues's theorem and explain how you could prove it without using duality.
3. Verify that the points on a Euclidean circle satisfy all the separation axioms except Axiom (ix), involving harmonic sets. Give an interpretation of harmonic sets of points on circle using harmonic sets of lines through the center. Does your interpretation satisfy Axiom (ix)?
4. Jacob Steiner defined conics using projectivities as follows. Let k_i be the family of lines on point P, let m_i be the family of lines on point Q, and assume that k_i is related to m_i by a projectivity of lines that isn't a perspectivity. Then the points R_i, which are the intersections of k_i and m_i, form a conic.
   a) Explore this method with graph paper. Let $P = (5, 12)$, $Q = (10, -7)$, lines k_i intersect the y-axis at $(0, i)$, and lines m_i intersect the x-axis at $(i, 0)$. Find various points R_i and sketch the conic.
   b) Repeat part (a) with $P = (-10, -6)$ and $Q = (6, 10)$.
   c) Identify the types of Euclidean conics you obtained in parts (a) and (b). Explain why points P and Q are always on the conic. Experiment with other placements of P and Q and other ways to relate the families of lines.
5. a) Use a dynamic geometry program to explore the following theorem in the special case of a circle. Pascal's theorem: Let A_1, A_2, A_3, A_4, A_5, and A_6 be any six points on a conic. Then the three points of intersections of the pairs of lines $\overleftrightarrow{A_1A_2} \cdot \overleftrightarrow{A_4A_5}$, $\overleftrightarrow{A_2A_3} \cdot \overleftrightarrow{A_5A_6}$, and $\overleftrightarrow{A_3A_4} \cdot \overleftrightarrow{A_6A_1}$ are collinear. (See Coxeter [2] for a proof.)
   b) Repeat part (a) where A_1, A_3, and A_5 are on one line and A_2, A_4, and A_6 are on another line.
   c) State and illustrate the dual of Pascal's theorem. (Pascal showed his theorem in 1640. Brianchon showed the dual in 1806. Only later did the concept of duality reveal the connection between these theorems.)
   d) State and illustrate the dual of part (b).
6. In Section 2.2 we defined an ellipse by using two foci. When we transform an ellipse to another ellipse by an affine transformation, do the old foci map to the new foci? Investigate with the ellipses $x^2/a^2 + y^2/b^2 = 1$ and affine transformations that map the x- and y-axes to themselves.
7. Relate homogeneous coordinates and barycentric coordinates. (See Section 2.3.)
8. A *correlation* is a transformation that maps points to lines and lines to points. Investigate correlations and how they relate to duality. (See Cederberg [1] or Coxeter [2].)
9. Investigate the possible types of eigenvalues and eigenvectors of a collineation. Find the possible sets of fixed points and stable lines. How does the set of fixed points relate to the set of stable lines for a collineation? [*Hint:* Use duality.] (See Fraleigh and Beauregard [3] for more on linear algebra.)
10. Investigate how computer-aided design uses projective geometry. (See Penna and Patterson [5].)
11. Investigate oriented projective geometry. (See Stolfi [7].)
12. Investigate the special theory of relativity. (See Taylor and Wheeler [8].)
13. Convert various ordinary equations, such as $y = x^2/(x^2 - 1)$, to homogeneous coordinates, in this case $x^2y - x^2z - yz^2 = 0$. Find the ideal points of these homogeneous equations and relate them to the graphs and asymptotes of the original equations. In algebraic geometry, usually a graduate-level subject, homogeneous coordinates are used to explore polynomial equations.
14. Write an essay discussing Klein's definition of geometry (see Section 4.2) in light of the various subgeometries of projective geometry.
15. Write an essay comparing the advantages and disadvantages of synthetic and analytic approaches to projective geometry. Do you agree with Poncelet that analytic geometry gives answers without insight? Explain.

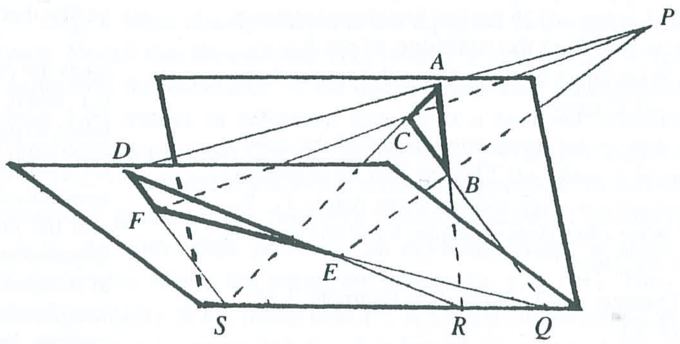

Figure 6.21 $\triangle ABC$ and $\triangle DEF$ are perspective from point P and from line $\overleftrightarrow{QR}$, where the planes containing the triangles intersect.

Suggested Readings

[1] Cederberg, J. *A Course in Modern Geometries*. New York: Springer-Verlag, 1989.

[2] Coxeter, H. *Projective Geometry*. Toronto: University of Toronto Press, 1974.

[3] Fraleigh, J., and R. Beauregard. *Linear Algebra*. Reading, Mass.: Addision-Wesley, 1987.

[4] Kline, M. *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press, 1972.

[5] Penna, M., and R. Patterson. *Projective Geometry and its Applications to Computer Graphics*. Englewood Cliffs, N.J.: Prentice Hall, 1986.

[6] Powell, W. *Perspective*. Tustin, Calif.: Walter Foster, 1989.

[7] Stolfi, J. *Oriented Projective Geometry*. Boston: Academic Press, 1991.

[8] Taylor, E., and J. Wheeler. *Spacetime Physics*. New York: W. H. Freeman, 1966.

[9] Tuller, A. *A Modern Introduction to Geometries*. New York: Van Nostrand Reinhold, 1967.

Suggested Media

1. "The Art of Renaissance Science: Galileo and Perspective," video, American Mathematical Society, Providence, 1991.
2. "Central Perspectivities," $13\frac{1}{2}$-minute film, International Film Bureau, Chicago, 1971.
3. "Conics in Perspective," 24-minute video, Films for the Humanities and Sciences, Princeton, N.J., 1996.

4. "Points of View: Perspective and Projection," 25-minute film, University Media, Solana Beach, Calif., 1975.
5. "Poles and Polars," $4\frac{1}{2}$-minute film, Educational Solutions, New York, 1979.
6. "Principles and Methods of Direct Perspective," 52-minute video, Pepper Publications, Tucson, 1987.
7. "Scientific Images in the Renaissance," 52-minute video, Churchill Films, Los Angeles, 1986.