to 1, and 2 to $x \neq \frac{1}{2}$. Verify that the harmonic set $H(0\,1, 2\,\frac{2}{3})$ goes to $H(0\,1, x\,\frac{x}{2x-1})$. [*Hint:* Pick $a = x$.]

d) Find the projectivity $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ that takes 0 to 0, 1 to ∞, and 2 to k. Find the image of $\frac{2}{3}$. Explain why this image forms a harmonic set with 0, ∞, and k.

3. Prove Theorem 6.4.3. [*Hint:* See Theorem 4.3.1.]

4. Prove Theorem 6.4.4.

5. Prove Theorem 6.4.5. [*Hint:* See Theorem 6.4.1.]

6. Recall Desargues's theorem from Section 6.1: If $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a point P, then they are perspective from a line. Let $A = (1, 0, 1)$, $B = (1, 1, 1)$, $C = (0, 1, 1)$, $A' = (2, 0, 1)$, $B' = (4, 4, 1)$, $C' = (0, 3, 1)$, and $P = (0, 0, 1)$.

a) Graph these points. (See Example 1, Section 6.3.) Find the intersections $\overleftrightarrow{AB} \cdot \overleftrightarrow{A'B'}$, $\overleftrightarrow{AC} \cdot \overleftrightarrow{A'C'}$, and $\overleftrightarrow{BC} \cdot \overleftrightarrow{B'C'}$ and verify that these points are collinear.

b) Find the collineation $\alpha = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$ that fixes P and takes A to A', B to B', and C to C'. [*Hint:* pick $a = 12$.] Verify that the line you found in part (a) is stable under α. Show further that every point on this line is fixed by α.

c) Make a conjecture generalizing part (b) and relate your conjecture to Desargues' theorem.

7. a) Let $M = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1/2 & -1/2 \end{bmatrix}$. Find M^{-1} and the image of the unit circle $x^2 + y^2 - z^2 = 0$ under M.

b) Explain why $[\cos\theta, \sin\theta, -1]$ are lines tangent to the unit circle.

c) Convert the image in part (a) to nonhomogeneous coordinates ($z = 1$). What type of Euclidean conic is this image?

d) (Calculus) Use derivatives to show that $[4k, -1, -2k^2]$ is tangent to the conic in part (c). Show that the ideal line $[0, 0, 1]$ is also a tangent.

8. If line k is tangent to conic C and α is any collineation, prove that the image of k under α is tangent to the image of C under α. [*Hint:* See Problem 12 of Section 6.3.]

9. Investigate the effect of the family of collineations $M_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ w & 0 & 1 \end{bmatrix}$ on the conic $C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (the circle $x^2 + y^2 - 2xz = 0$).

a) Find M_w^{-1} and M_w^{-1T} and the general image $M_w^{-1T} C M_w^{-1}$ of C under M_w.

b) For the following values of w, convert $M_w^{-1T} C M_w^{-1}$ to the equation of a conic in nonhomogeneous coordinates ($z = 1$): $w = \frac{1}{2}$, $-\frac{1}{4}$, $-\frac{1}{2}$, and -1.

c) Graph the original circle and the conics of part (b).

d) Which values of w in part (a) take the circle to Euclidean ellipses? to hyperbolas? to parabolas?

e) Verify that the lines $[1, 0, -2]$ and $[0, 1, -1]$ are tangent to the original circle by graphing them with the circle.

f) Find the images of the lines of part (e) for the values of w in part (b) and graph them with the corresponding conics to verify that they are tangents.

6.5 Subgeometries

Projective geometry originated as an extension of Euclidean geometry. In 1858 Arthur Cayley provided the historically important construction of Euclidean distance and angle measure within projective geometry. That is, he showed Euclidean geometry to be a subgeometry of projective geometry. He also related the distance in spherical geometry to projective geometry but was unaware of any other geometries at that time. Felix Klein in 1871 built on Cayley's construction and other insights to show that both hyperbolic and single elliptic geometries (see Chapter 3) were subgeometries of projective geometry. Klein's construction led him to pick the names hyperbolic and elliptic.

ARTHUR CAYLEY

The most fundamental mathematical contributions of Arthur Cayley (1821–1895) came during the 15 years he practiced law following his mathematical education at Cambridge University. Cayley was first in his class and started teaching there, but soon left because he didn't want to be ordained as a priest, then required of all Cambridge professors. Later, as a renowned mathematician, he taught at Cambridge (without ordination) from 1863 until his death, except for a year in the United States. Cayley published nearly two hundred papers while practicing law and hundreds more in his lifetime, largely on algebra and geometry.

At the age of 20, Cayley started publishing on algebraic invariants, a subject now displaced by topics in algebra and geometry that he and others developed from invariants. While in his early 20s he published some of the first works on n-dimensional geometry. In 1845, at age 24, he found an 8-dimensional algebra generalizing the complex numbers and Hamilton's recently published 4-dimensional quaternions. In 1849, he proposed the abstract definition of a group and later published the theorem on groups named after him.

Algebraists had studied determinants for some time, but Cayley's work on invariants led him to be the first, in 1855, to study matrices and their properties. He used matrices to represent systems of equations and transformations. He defined matrix multiplication so that it would correspond to the composition of transformations. His crowning achievement in what we now call linear algebra came in 1858 with publication of the Cayley–Hamilton theorem on the characteristic equation of a matrix. Invariants also led Cayley in the following year to his key derivation of Euclidean geometry within projective geometry relative to an absolute conic. He had long used projective geometry in his study of algebraic invariants, which included conics as second-degree invariants. His method of finding metrical properties inside projective geometry led to the unification of several geometries. Cayley learned of hyperbolic geometry after publishing this important paper, but he never accepted it as more than a logical curiosity.

(Under his classification Euclidean geometry was a "parabolic" geometry.) The unification of many geometries under projective geometry emphasized the usefulness of projective geometry and the importance of modern, abstract mathematics.

One geometry is a subgeometry of another in two regards. First, from Klein's Erlanger Programm (see Section 4.2), if all transformations of one geometry are transformations of a second geometry, the first is a subgeometry of the second. Second, the points, lines, and other terms of the subgeometry must be defined from the points, lines, and the like of the encompassing geometry. Fortunately, these two conditions reinforce each other. Figure 6.12 shows relationships of various geometries in terms of their transformation groups.

Recall that projective properties of points on a line must involve at least four points because any three points can be mapped to any three points by Theorem 6.2.7. Cross ratios, harmonic sets, and separation, involving four points, are preserved under all collineations by Theorem 6.4.2, as are projective properties. However, distance is a relation involving just two points. Cayley realized that he could define the distance between two points relative to two other points on a given line. To obtain these other

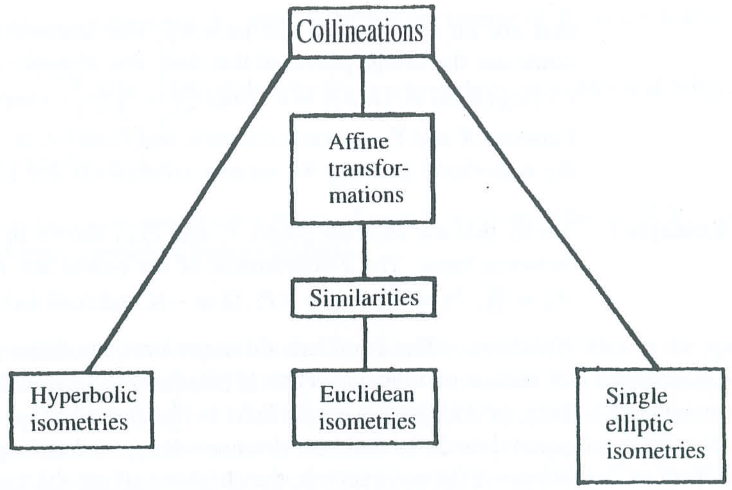

Figure 6.12 Some subgroups of collineations.

points he used the intersection of the line with a fixed conic, which he called the *absolute conic*. Cayley similarly defined the angle between two lines by using two other concurrent lines tangent to the absolute conic. We develop this idea fully only for distance in Klein's model of hyperbolic geometry. The other situations are more complicated, for they involve complex numbers and other advanced concepts. The transformations for various geometries are characterized as projective collineations that take the absolute conic to itself. (See Tuller [9, Chapter 7] for more information.)

6.5.1 Hyperbolic geometry as a subgeometry

From a projective point of view, in the Klein model of hyperbolic geometry in Chapter 3 we could use the points in the interior of any (real, nondegenerate) conic, but the unit circle is the simplest. In the following interpretation, this conic is the absolute conic in Cayley's terms. Note that any projective line through an interior point intersects the conic in two points. Indeed, one defines a point to be in the *interior* of a conic if every line on that point intersects the conic in two distinct real points. These two points of intersection of a line and the conic match the property that hyperbolas intersect the ideal line in two points. Klein used this analogy and other reasons to call this geometry hyperbolic. He based the formula for hyperbolic distance on the cross ratio of the two points and the intersections of the line they determine with the conic. Recall that all collineations preserve the cross ratio, so automatically Klein's distance formula is preserved under whichever collineations are hyperbolic. Poincaré defined hyperbolic distance in his model in the same way we do here. (See Sections 3.4 and 4.6.) Poincaré recognized that the cross ratio is preserved under both inversions and collineations. To simplify the distance formula we include a Euclidean way to compute it.

Interpretation for Hyperbolic Plane Geometry By *absolute conic* we mean $x^2 + y^2 - z^2 = 0$. By *point* we mean the points (x, y, z), with $x^2 + y^2 < z^2$, interior to the absolute conic. By *line* we mean the set of points interior to the absolute conic that are on a projective line $[a, b, c]$. The intersections of a line with the absolute conic are the *omega points* of that line. The *distance* between A and B is $d_H(A, B) = c \cdot |\log(R(A, B, \Omega, \Lambda)| = c \cdot \left|\log(\frac{A\Omega}{A\Lambda} \div \frac{B\Omega}{B\Lambda})\right|$, where XY is the Euclidean distance between X and Y, c is some constant, and Ω and Λ are the two omega points of line $\overleftrightarrow{AB}$. By *hyperbolic isometry* we mean a collineation that leaves the absolute conic stable.

Example 1 Verify that the adjacent points P_i and P_{i+1} shown in Fig. 6.13 have the same distance between them. The x-coordinates of the points are $P_0 = 0$, $P_1 = \frac{1}{3}$, $P_2 = \frac{3}{5}$, $P_3 = \frac{7}{9}$, $P_4 = \frac{15}{17}$, $P_5 = \frac{31}{33}$, $P_{-i} = -P_i$, $\Omega = -1$, and $\Lambda = 1$.

Solution. The Euclidean distances between these points are simply the differences of their x-coordinates. Then $(P_0\Omega/P_0\Lambda) \div (P_1\Omega/P_1\Lambda) = (1/1) \div (\frac{4}{3}/\frac{2}{3}) = \frac{1}{2}$. Similarly, $(P_1\Omega/P_1\Lambda) \div (P_2\Omega/P_2\Lambda) = (\frac{4}{3}/\frac{2}{3}) \div (\frac{8}{5}/\frac{2}{5}) = \frac{1}{2}$. All the corresponding products equal $\frac{1}{2}$ or 2. In turn, the absolute values of their logarithms are all the same. Hence, whatever the constant c is, the distances all are the same. ●

Example 2 Euclidean rotations and mirror reflections fixing the origin map the unit circle to itself and so are hyperbolic isometries. A matrix X_x analogous to a translation shifting $(0, 0, 1)$ along the x-axis to $(x, 0, 1)$ has for its last column $(x, 0, 1)$, with $-1 < x < 1$. This translation leaves the omega points $(1, 0, 1)$ and $(-1, 0, 1)$ fixed. Then $X_x = \begin{bmatrix} a & d & x \\ b & e & 0 \\ c & f & 1 \end{bmatrix}$ must satisfy $X_x(1, 0, 1) = (a + x, b, c + 1) = \lambda(1, 0, 1)$ and $X_x(-1, 0, 1) = (-a + x, -b, -c + 1) = \lambda(-1, 0, 1)$. From these equations $a = 1$, $b = 0$, and $c = x$. Figure 6.14 suggests that X_x should shift $(0, 1, 1)$ and $(0, -1, 1)$ to the points on the unit circle directly above and below $(x, 0, 1)$. This condition forces $d = 0 = f$ and $e = \sqrt{1 - x^2}$. ●

Exercise 1 Let $X_x = \begin{bmatrix} 1 & 0 & x \\ 0 & \sqrt{1-x^2} & 0 \\ x & 0 & 1 \end{bmatrix}$. Verify that $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, the unit circle, is stable under any scalar multiple of X_x, and verify that the inverse of X_x is a scalar multiple of $\begin{bmatrix} 1 & 0 & -x \\ 0 & \sqrt{1-x^2} & 0 \\ -x & 0 & 1 \end{bmatrix}$. Explain why X_x is a hyperbolic isometry and why, in effect, X_{-x} is the inverse of X_x.

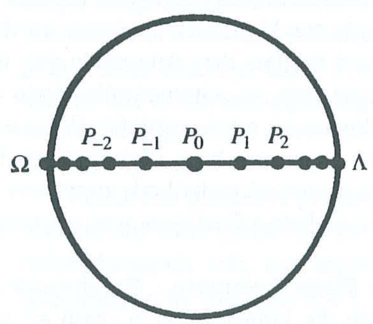

Figure 6.13

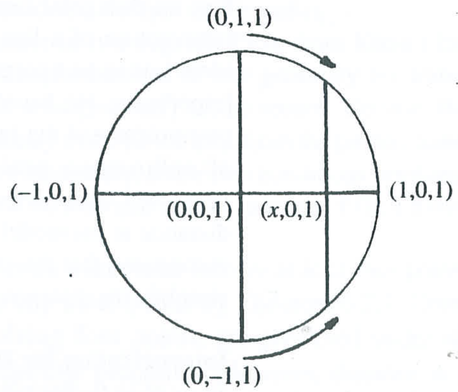

Figure 6.14

Theorem 6.5.1 The hyperbolic isometries form a group of transformations. The set $\{\lambda X_x : -1 < x < 1, \lambda \neq 0\}$ forms a group of transformations.

Proof. See Problem 3. ■

The matrices X_x correspond to the transformations of velocities in the special theory of relativity. (See Section 5.5.) In Section 6.6 we explore this connection, including representing the geometry of relativity as a subgeometry of four-dimensional projective space. General hyperbolic isometries have a matrix form quite similar to spherical isometries. (See Section 4.5.) A hyperbolic isometry M must take C to itself. By Theorem 6.4.3, $M^{-1T}CM^{-1} = \lambda C$, for $\lambda \neq 0$. The special form of C simplifies this equation further. To emphasize the relationship of these isometries with spherical isometries, define the *h-inner product* of two vectors to be $(r, s, t) \cdot_h (u, v, w) = ru + sv - tw$. Only the minus sign distinguishes this h-inner product from the usual definition. Following this analogy, define the *h-length* of a vector (r, s, t) to be $(r, s, t) \cdot_h (r, s, t)$ and two vectors to be *h-orthogonal* iff their h-inner product is 0. (These definitions do not fulfill all the usual properties of inner products and lengths. For example, nonzero vectors can be h-orthogonal to themselves and so have zero h-lengths.)

Example 3 Find conditions on M so that M is a hyperbolic isometry.

Solution. For M to be a hyperbolic isometry, we must have $M^{-1T}CM^{-1} = \lambda C$ for some $\lambda \neq 0$. Multiply both sides of this equation by M^T on the left and M on the right to get $C = M^T\lambda CM$. We can factor out λ and move it to the other side to get $\frac{1}{\lambda}C = M^TCM$. If we write P for the first column, Q for the second column and R for the third column of M, then $M^TCM = M^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} M = \begin{bmatrix} P \cdot_h P & P \cdot_h Q & P \cdot_h R \\ Q \cdot_h P & Q \cdot_h Q & Q \cdot_h R \\ R \cdot_h P & R \cdot_h Q & R \cdot_h R \end{bmatrix}$, which is supposed to equal $\frac{1}{\lambda}C = \begin{bmatrix} 1/\lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & -1/\lambda \end{bmatrix}$. Then the columns of M must be h-orthogonal to each other to give 0 off the main diagonal. Furthermore, the first two columns must have the same h-length, which must be the negative of the h-length of the third column. ●

Exercise 2 Verify that the X_x satisfy the conditions of Example 3.

6.5.2 Single elliptic geometry as a subgeometry

Single elliptic geometry (see Section 3.5) needs all projective points and lines. Thus the absolute conic must contain no (real) points. For the absolute conic we pick the imaginary conic $x^2 + y^2 + z^2 = 0$, whose only real solution is $(0, 0, 0)$, which isn't a projective point. Lines in single elliptic geometry therefore don't intersect the absolute conic, just as an ellipse has no intersection with the ideal line, suggesting one reason why Klein called this geometry elliptic. (He added single to the name to distinguish it from spherical geometry, in which lines intersect twice.) An isometry M for this geometry is a collineation that takes this degenerate conic $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to itself. Because $C = I$, the identity, the equation $M^{-1T}CM^{-1} = \lambda C$ for $\lambda \neq 0$ reduces to $M^{-1T}M^{-1} = \lambda I$. Orthogonal matrices, the isometries for spherical geometry (see Section 4.5), satisfy the similar equation $M^T = M^{-1}$, or $M^T M = I$.

Theorem 6.5.2 A collineation M is an isometry of single elliptic geometry iff $M^T = \lambda M^{-1}$ for some nonzero real number λ. These isometries form a group of transformations.

Proof. See Problem 5. ∎

6.5.3 Affine and Euclidean geometries as subgeometries

Recall that the projective plane can be thought of as the affine or Euclidean plane and one extra "ideal" line, [0, 0, 1]. Cayley realized that [0, 0, 1], thought of as the degenerate conic $z^2 = 0$, functioned as the absolute conic. As affine geometry has no notions of distance or angle measure, we can directly give the interpretation of the affine plane as a subgeometry of projective geometry. Note that each affine line intersects the absolute conic in one point just as a parabola intersects the ideal line in one point. Klein considered affine geometry and its subgeometries to be parabolic.

Interpretation for Affine Plane Geometry By *absolute conic* we mean $z^2 = 0$. By *point* we mean a point not on the absolute conic: $(x, y, 1) = (\lambda x, \lambda y, \lambda)$. By *line* we mean a line other than the absolute conic: $[a, b, c] = [\lambda a, \lambda b, \lambda c]$, where not both a and b are zero. An *affine transformation* is a collineation that leaves the absolute conic stable.

Exercise 3 Verify that the affine transformations of Chapter 4 are the affine transformations of the preceding interpretation.

To derive Euclidean distance, Cayley needed to pick two specific points on the absolute conic $z^2 = 0$, the *circular points at infinity*, or $I = (1, i, 0)$ and $J = (1, -i, 0)$. Their name comes from the fact, noted by Poncelet, that these points are on every Euclidean circle. Example 4 reveals that all Euclidean isometries map I and J to themselves, showing the set $\{I, J\}$ to be stable under this group of transformations. The cross-ratio definition of distance (and angle measure) Klein employed in hyperbolic and single elliptic geometries works for Euclidean angle measure but not for Euclidean distance. Cayley utilized a more complicated method, which we omit.

Example 4 Show that all Euclidean isometries map I and J to themselves.

Solution. Recall from Section 4.3 that Euclidean isometries are of the form $\begin{bmatrix} \cos\theta & -\sin\theta & c \\ \sin\theta & \cos\theta & f \\ 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} \cos\theta & \sin\theta & c \\ \sin\theta & -\cos\theta & f \\ 0 & 0 & 1 \end{bmatrix}$. The first form, for direct isometries, takes $I = (1, i, 0)$ to $(\cos\theta - i\sin\theta, \sin\theta + i\cos\theta, 0) = (\cos\theta - i\sin\theta, i(\cos\theta - i\sin\theta), 0) = (1, i, 0)$. Similarly, J maps to itself. Matrices of the second form, for indirect isometries, switch I and J. ●

PROBLEMS FOR SECTION 6.5

1. a) Verify that the coordinate for each P_i in Example 1 is $(2^i - 1)/(2^i + 1)$.
   b) Verify that $X_{1/3}$, as defined in Exercise 1, takes P_i to P_{i+1}.
   c) Verify that $X_{1/3}$, as defined in Exercise 1, takes $-P_{i+1}$ to $-P_i$.
   d) Verify that $X_{1/3} \cdot X_{1/3}$ is a scalar multiple of $X_{3/5}$.
2. For the points $A = (a, 0, 1)$ and $B = (b, 0, 1)$, verify the identity $d_H(A, B) = d_H(X_x(A), X_x(B))$, where X_x is the hyperbolic translation of Exercise 1. [*Hint:* Simplify the cross-ratio for $X_x(A)$, $X_x(B)$, $\Omega = (1, 0, 1)$, and $\Lambda = (-1, 0, 1)$ by factoring.]
3. a) Prove that the composition of the hyperbolic translations X_a and X_b is a scalar multiple of $X_{a\oplus b}$, where $a \oplus b = (a + b)/(1 + ab)$ is the addition of velocities in the special theory of relativity. (See Section 5.5.)
   b) Prove Theorem 6.5.1. [*Hint:* See Problem 7 of Section 5.5.)
4. Find the matrix for a hyperbolic translation Y_b along the y-axis. Verify that, in general, $X_a \cdot Y_b \neq Y_b \cdot X_a$. How are $X_a \cdot Y_b$ and $Y_b \cdot X_a$ related as matrices? Let $(c, d, 1)$ be any point inside the unit circle. Find hyperbolic translations X_a and Y_b such that $X_a \cdot Y_b(0, 0, 1) = (c, d, 1)$.
5. Prove Theorem 6.5.2. [*Hint:* See Theorem 4.5.1.]
6. a) Show that a conic $ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 = 0$ has the points $(1, i, 0)$ and $(1, -i, 0)$ on it iff it is a Euclidean circle; that is, $a = c$ and $b = 0$.
   b) Show that all similarities take the circular points at infinity to themselves.
7. Find the properties of an inner product and of length in a linear algebra text and explain which ones hold and which ones fail for the h-inner product and the h-length. (See, for example, Fraleigh and Beauregard [3].)
8. a) Show that every single elliptic isometry that is simultaneously an affine transformation must be a similarity. Are there any other restrictions on these transformations? [*Hint:* What are the images of the points $O = (0, 0, 1)$, $X_1 = (1, 0, 1)$, and $Y_1 = (0, 1, 1)$ under a similarity?]
   b) Describe all collineations that are simultaneously single elliptic isometries, hyperbolic isometries, and Euclidean isometries. Justify your answer and show that these collineations form a group.

6.6 Projective Space

Homogeneous coordinates and collineations can be readily extended to higher dimensional projective spaces, which are significant models that are used for many purposes. For example, three-dimensional projective space provides perspective views for computer-aided design (CAD). And the Lorentz transformations in the special theory of relativity can be seen as isometries in a subgeometry of four-dimensional projective space related to hyperbolic geometry.

Interpretation By a *point* in n-dimensional projective space $\mathbf{P}^n$ we mean a one-dimensional subspace of the $n + 1$ dimensional vector space $\mathbf{R}^{n+1}$. By *line* we mean a two-dimensional