10. (Calculus) **a)** Write the equation $y = x^2$ in homogeneous coordinates.

b) Show that the lines $[a, b, c]$ tangent to the equation in part (a) satisfy the equation of the line conic $a^2 - 4bc = 0$ as follows. First, pick x_0, a real number, and find the coordinates of the point $(x_0, y_0, 1)$ on the conic. Next use calculus to find the equation of the tangent line. Now convert this line to the form $[a, b, c]$ and verify that it satisfies the equation $a^2 - 4bc = 0$.

c) Graph $y = x^2$ and selected tangents.

11. (Calculus) Repeat Problem 10 for the hyperbola $y = \frac{1}{x}$ and the corresponding line conic $4ab - c^2$. Explain the similarities in these problems from a projective viewpoint.

12. Suppose that P is a point on the conic C. Claim: The line $l = P^T C$ is tangent to C at P.

a) Verify the claim by using the matrix form for conics in Problems 10 and 11.

b) Show that P is on the line l of the claim.

c) Suppose that P and Q are two points on the conic C and that l and k are their corresponding lines in the claim. Show that $l = k$ iff $P = Q$. Explain why this equation shows that the lines in the claim intersect the conic in only one point. Write a proof of the claim.

d) Show that all tangents l satisfy the equation of the line conic $lC^{-1}l^T = 0$.

6.4 PROJECTIVE TRANSFORMATIONS

The usefulness of homogeneous coordinates becomes apparent in terms of transformations. Recall from Chapter 4 that a transformation is a one-to-one function from a space onto itself and that points are column vectors and lines are row vectors. Thus projectivities (transformations of collinear points) and collineations (transformations of all the points of the projective plane) correspond to invertible matrices. If we use two homogeneous coordinates for points on a line, a projectivity maps points (u, v) to points (s, t) and can be represented as an invertible 2×2 matrix. This matrix can be considered to map a line to itself.

Interpretation A *projectivity* is represented by an invertible 2×2 matrix. Two such matrices that differ by a nonzero constant represent the same projectivity.

Problem 2 of Section 6.1 shows that a projectivity can map any three collinear points to any three collinear points. Theorem 6.4.1 shows the surprising fact that 2×2 matrices can mimic this flexibility.

Theorem 6.4.1 For any distinct collinear points (u_1, v_1), (u_2, v_2), and (u_3, v_3) and distinct collinear points (s_1, t_1), (s_2, t_2), and (s_3, t_3), a unique projectivity maps (u_i, v_i) to (s_i, t_i).

Proof. First we show that a projectivity can map the three collinear points $(1, 0)$, $(0, 1)$, and $(1, 1)$ to any three distinct collinear points (s_1, t_1), (s_2, t_2), and (s_3, t_3). The matrix $\begin{bmatrix} s_1 & s_2 \\ t_1 & t_2 \end{bmatrix}$ maps $(1, 0)$ to (s_1, t_1) and $(0, 1)$ to (s_2, t_2). If we multiply the first column of the matrix by any scalar multiple λ, $(1, 0)$ still goes to $(s_1, t_1) = (\lambda s_1, \lambda t_1)$ and similarly for the second column, giving us the needed flexibility. For the matrix $\begin{bmatrix} \lambda s_1 & \mu s_2 \\ \lambda t_1 & \mu t_2 \end{bmatrix}$ the image of $(1, 1)$ is $(\lambda s_1 + \mu s_2, \lambda t_1 + \mu t_2)$, which we set equal to (s_3, t_3) and solve for λ and μ. Because (s_2, t_2) is not a multiple of (s_1, t_1) the determinant $\begin{vmatrix} s_1 & s_2 \\ t_1 & t_2 \end{vmatrix} \neq 0$. Thus there is a solution. In fact, the solutions form a family that are scalar multiples of one another. Thus a unique projectivity M takes $(1, 0)$, $(0, 1)$, and $(1, 1)$ to any three distinct points (s_1, t_1), (s_2, t_2), and (s_3, t_3). Similarly, a unique projectivity K takes $(1, 0)$, $(0, 1)$, and $(1, 1)$ to (u_1, v_1), (u_2, v_2), and (u_3, v_3). Then the projectivity represented by MK^{-1} uniquely satisfies the theorem's conditions. ■

Theorem 6.4.2 Cross ratios, harmonic sets, and separation are preserved under projectivities.

Proof. Let (u_i, v_i) for $i = 1, 2, 3, 4$ be four collinear points and $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ be any projectivity. We first show the cross ratios $R(u_1, u_2, u_3, u_4)$ and $R(Mu_1, Mu_2, Mu_3, Mu_4)$ to be equal. A cross ratio involves four determinants of the coordinates of the points. For example, consider $\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}$ and the corresponding determinant of the images $M(u_1, v_1)$ and $M(u_3, v_3)$: $\begin{vmatrix} Mu_1 & Mu_3 \\ Mv_1 & Mv_3 \end{vmatrix} = |M| \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}$. The extra factor of $|M|$ appears in each of the four determinants in the cross ratio of the four images. As two of these factors of $|M|$ are in the numerator and two in the denominator, they cancel each other. Thus $R(u_1, u_2, u_3, u_4) = R(Mu_1, Mu_2, Mu_3, Mu_4)$. Harmonic sets and separation are defined in terms of the cross ratio, so they also are preserved under projectivities. ■

Interpretation A *collineation* of the projective plane is represented by an invertible 3×3 matrix. Two matrices represent the same collineation iff one is a nonzero scalar multiple of the other. A point P is *fixed* by a collineation α iff $\alpha(P) = P$. A line k is *stable* under a collineation α iff every point on k is mapped by α to a point on k.

All the affine transformations from Chapter 4, including isometries and similarities, are collineations because all affine transformations have invertible matrices of the form $\begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix}$. Theorem 6.4.3 describes how to find the images of lines and conics under collineations. Although points, lines, conics, and collineations have multiple representations, for simplicity we use just one such representation in the examples. In essence a collineation is a projectivity for each line, so Theorem 6.4.2 applies to both collineations and projectivities.

Theorem 6.4.3 The image of the line $[a, b, c]$ under the collineation M is $[a, b, c]M^{-1}$. The image of the conic C under M is $M^{-1T}CM^{-1}$.

Proof. See Problem 3. ■

Theorem 6.4.4 The set of projectivities of a line to itself forms a group of transformations. The set of collineations forms a group of transformations.

Proof. See Problem 4. ■

Example 1 Consider the Euclidean translation $T = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. In Chapter 4 we showed that a translation has no fixed affine points $(x, y, 1)$. However, we now have more points (x, y, z). Because $\lambda(x, y, z)$ represents the same point as (x, y, z), we need to consider the general eigenvector problem $T(x, y, z) = \lambda(x, y, z)$. Note that eigenvectors for any nonzero eigenvalue represent fixed points. We obtain three equations: $x + 3z = \lambda x$, $y - 2z = \lambda y$, and $z = \lambda z$. The last equation forces $\lambda = 1$ or $z = 0$. In turn, the first two equations force both $\lambda = 1$ and $z = 0$. Thus every "ideal point" $(x, y, 0)$ is fixed by a translation. Recall that ideal points are where Euclidean parallel lines meet and that translations take a line to a line parallel to itself. We also showed that all the stable lines of a translation in the affine plane are parallel, in this case $[-\frac{2}{3}, -1, c]$. As a collineation, there is one more stable line: $[0, 0, 1]$, the ideal line $z = 0$. All these stable lines can be written in the form $[\frac{-2}{3}a, -a, c]$ and go through the point $(3, -2, 0)$. ●

Example 2 The ideal line $[0, 0, 1]$ is stable for all similarities and affine transformations, for the bottom row of each of their inverses is $[\,0 \quad 0 \quad 1\,]$. Consider $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which reflects over the line $y = x - z$ ($[1, -1, -1]$) and expands by a factor of 2 around the fixed point $(-1, -2, 1)$. There are two other fixed projective points, which we find by solving the general eigenvector problem $A(x, y, z) = \lambda(x, y, z)$. There are three values of λ such that the determinant of $A - \lambda I$ is zero: 1, 2, and -2. The three fixed points are $(-1, -2, 1)$, $(1, 1, 0)$, and $(1, -1, 0)$, respectively. To find the stable lines we need to solve the general eigenvector problem $[a, b, c]A^{-1} = \lambda[a, b, c]$. Verify that $A^{-1} = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & -1.5 \\ 0 & 0 & 1 \end{bmatrix}$ and that the three eigenvalues are 1, 0.5, and -0.5, the multiplicative inverses of the eigenvalues of A. The corresponding stable lines are $[0, 0, 1]$, $[-1, -1, -3]$, and $[1, -1, -1]$. Verify that these lines are stable and intersect in the three fixed points. ●

Because collineations are 3×3 matrices, we can send the points $X = (1, 0, 0)$, $Y = (0, 1, 0)$, and $Z = (0, 0, 1)$ to any three noncollinear points. However, as in Theorem 6.4.1, we can do even more. We can send X, Y, Z, and $U = (1, 1, 1)$ to any four points, provided no three of them are collinear.

Exercise 1 Verify that the collineation $\begin{bmatrix} 6 & -1 & 0 \\ 9 & 1 & 0 \\ 3 & -1 & 3 \end{bmatrix}$ takes X, Y, Z, and U to $(2, 3, 1)$, $(1, -1, 1)$, $(0, 0, 1)$, and $(1, 2, 1)$, respectively.

Theorem 6.4.5 Let P_1, P_2, P_3, and P_4 be four points, no three of which are collinear, and Q_1, Q_2, Q_3, and Q_4 be any four points. Then a unique collineation takes P_1, P_2, P_3, and P_4 to Q_1, Q_2, Q_3, and Q_4, respectively, iff no three of Q_1, Q_2, Q_3, and Q_4 are collinear.

Proof. See Problem 5. ■

In Chapter 4 we were able to distinguish different types of isometries by the fixed points and stable lines of each. For example, rotations of any angle other than 180° and 0° have one fixed point and no stable lines. Translations have no fixed points and a family of parallel stable lines. Projective geometry has an additional line and the points on it, which radically alter the existence of fixed points and stable lines. As collineations, all isometries and, more generally, all affine transformations leave stable this added line. Translations fix every point on this added line. Theorem 6.4.6 relies on linear algebra to ensure the existence of fixed points and stable lines.

Theorem 6.4.6 Every collineation of the projective plane has at least one fixed point and at least one stable line.

Proof. To solve the general eigenvector problem $A(x, y, z) = \lambda(x, y, z)$ we find the values of λ for which the determinant of $A - \lambda I$ is zero. As $A - \lambda I$ is a 3×3 matrix, with λ appearing in the three diagonal entries, the determinant (characteristic polynomial) is a third-degree real polynomial in λ. Every third-degree polynomial crosses the x-axis and so has at least one real root. Thus every collineation has an eigenvalue and a nonzero eigenvector, which is a fixed point. The same argument applies to the inverse matrix, giving a stable line. ■

Example 3 The matrix $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is a rotatory reflection when thought of as a spherical isometry or three-dimensional Euclidean isometry. It rotates the sphere 90° around the z-axis and reflects it over the equator (xy-plane). Note that as it is a spherical isometry, no point is fixed. However, $(0, 0, 1)$ is mapped to $(0, 0, -1)$, which is the same projective point. This fixed point is for the only real eigenvalue, -1. The only stable line is $[0, 0, 1]$, the equator or the ideal line. ●

Recall from Section 6.3 that a conic is determined by five points, no three collinear. Theorem 6.4.5 asserts that collineations are determined by four points and their images. Surprisingly, despite this disparity, we can map every conic to every other conic, although we can't always specify where various points on one conic map onto the other conic. Problem 9 illustrates this flexibility.

PROBLEMS FOR SECTION 6.4

1. a) Describe all 2×2 matrices that send $(1, 0)$ to itself. Repeat for the point $(0, 1)$ and the point $(1, 1)$. Use these results to explain why the only 2×2 matrices leaving these points fixed are of the form $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

b) Repeat part (a) with 3×3 matrices and the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$.

2. Let $x = (x, 1)$ and $\infty = (1, 0)$. Recall from Section 6.1 that $H(0\ 1, x\ \frac{x}{2x-1})$.

a) Find the projectivity $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ that takes 0 to 0, 1 to 1, and 2 to $\frac{3}{2}$. Verify that the harmonic set $H(0\ 1, 2\ \frac{2}{3})$ goes to a harmonic set.

b) Find the projectivity $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ that takes 0 to 0, 1 to 1, and 2 to $\frac{1}{2}$. Why does the harmonic set $H(0\ 1, 2\ \frac{2}{3})$ go to a harmonic set?

c) Find the projectivity $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ that takes 0 to 0, 1 to 1, and 2 to $x \neq \frac{1}{2}$. Verify that the harmonic set $H(0\ 1, 2\ \frac{2}{3})$ goes to $H(0\ 1, x\ \frac{x}{2x-1})$. [*Hint:* Pick $a = x$.]

d) Find the projectivity $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ that takes 0 to 0, 1 to ∞, and 2 to k. Find the image of $\frac{2}{3}$. Explain why this image forms a harmonic set with 0, ∞, and k.

3. Prove Theorem 6.4.3. [*Hint:* See Theorem 4.3.1.]

4. Prove Theorem 6.4.4.

5. Prove Theorem 6.4.5. [*Hint:* See Theorem 6.4.1.]

6. Recall Desargues's theorem from Section 6.1: If $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a point P, then they are perspective from a line. Let $A = (1, 0, 1)$, $B = (1, 1, 1)$, $C = (0, 1, 1)$, $A' = (2, 0, 1)$, $B' = (4, 4, 1)$, $C' = (0, 3, 1)$, and $P = (0, 0, 1)$.

a) Graph these points. (See Example 1, Section 6.3.) Find the intersections $\overleftrightarrow{AB} \cdot \overleftrightarrow{A'B'}$, $\overleftrightarrow{AC} \cdot \overleftrightarrow{A'C'}$, and $\overleftrightarrow{BC} \cdot \overleftrightarrow{B'C'}$ and verify that these points are collinear.

b) Find the collineation $\alpha = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$ that fixes P and takes A to A', B to B', and C to C'. [*Hint:* pick $a = 12$.] Verify that the line you found in part (a) is stable under α. Show further that every point on this line is fixed by α.

c) Make a conjecture generalizing part (b) and relate your conjecture to Desargues' theorem.

7. a) Let $M = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1/2 & -1/2 \end{bmatrix}$. Find M^{-1} and the image of the unit circle $x^2 + y^2 - z^2 = 0$ under M.

b) Explain why $[\cos\theta, \sin\theta, -1]$ are lines tangent to the unit circle.

c) Convert the image in part (a) to nonhomogeneous coordinates ($z = 1$). What type of Euclidean conic is this image?

d) (Calculus) Use derivatives to show that $[4k, -1, -2k^2]$ is tangent to the conic in part (c). Show that the ideal line $[0, 0, 1]$ is also a tangent.

8. If line k is tangent to conic C and α is any collineation, prove that the image of k under α is tangent to the image of C under α. [*Hint:* See Problem 12 of Section 6.3.]

9. Investigate the effect of the family of collineations $M_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ w & 0 & 1 \end{bmatrix}$ on the conic $C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (the circle $x^2 + y^2 - 2xz = 0$).

a) Find M_w^{-1} and M_w^{-1T} and the general image $M_w^{-1T} C M_w^{-1}$ of C under M_w.

b) For the following values of w, convert $M_w^{-1T} C M_w^{-1}$ to the equation of a conic in nonhomogeneous coordinates ($z = 1$): $w = \frac{1}{2}$, $-\frac{1}{4}$, $-\frac{1}{2}$, and -1.

c) Graph the original circle and the conics of part (b).

d) Which values of w in part (a) take the circle to Euclidean ellipses? to hyperbolas? to parabolas?

e) Verify that the lines $[1, 0, -2]$ and $[0, 1, -1]$ are tangent to the original circle by graphing them with the circle.

f) Find the images of the lines of part (e) for the values of w in part (b) and graph them with the corresponding conics to verify that they are tangents.

6.5 SUBGEOMETRIES

Projective geometry originated as an extension of Euclidean geometry. In 1858 Arthur Cayley provided the historically important construction of Euclidean distance and angle measure within projective geometry. That is, he showed Euclidean geometry to be a subgeometry of projective geometry. He also related the distance in spherical geometry to projective geometry but was unaware of any other geometries at that time. Felix Klein in 1871 built on Cayley's construction and other insights to show that both hyperbolic and single elliptic geometries (see Chapter 3) were subgeometries of projective geometry. Klein's construction led him to pick the names hyperbolic and elliptic.