In particular, do the duals of the separation axioms hold?

b) In analytic geometry, if m_1 and m_2 are the bisectors of the angles formed by m_3 and m_4, then $H(m_1m_2, m_3m_4)$. Investigate whether this relation holds for the labeling of the lines given, following the construction of a harmonic set of lines from Problem 7 of Section 6.1, with $m_1 = k$ and $m_2 = k_0$.

6.3 Analytic Projective Geometry

We develop homogeneous coordinates to represent analytically all the points and lines in projective geometry and to emphasize their duality. In addition, homogeneous coordinates enable us to consider projective transformations in Section 6.4. In Section 6.2 we showed that two coordinates aren't sufficient to describe all points in the projective plane. In Chapter 4 we used three coordinates $(x, y, 1)$ for points, enabling transformations to move all points. In the process we demonstrated that row vectors $[a, b, c]$ represent lines, where $(x, y, 1)$ is on $[a, b, c]$ provided that $ax + by + c \cdot 1 = 0$. In addition, we showed that nonzero multiples $[\lambda a, \lambda b, \lambda c]$ of $[a, b, c]$ represent the same line. The duality of points and lines in projective geometry suggests using triples (x, y, z) for the enlarged set of projective points. As in Chapter 4 we use (x, y, z) for the column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Examples 1, 2, and 3 provide three ways to view these points. Example 3 is the most important representation. We leave it as an exercise to verify that the following interpretation satisfies the first three axioms of Section 6.2.

Interpretation By a *point* in the projective plane, we mean a nonzero column vector (x, y, z), where (x, y, z) and $(\lambda x, \lambda y, \lambda z)$ represent the same point for $\lambda \neq 0$. By a *line* in the projective plane, we mean a nonzero row vector $[a, b, c]$, where $[a, b, c]$ and $[\lambda a, \lambda b, \lambda c]$ represent the same line for $\lambda \neq 0$. A point (x, y, z) is *on* the line $[a, b, c]$ iff $ax + by + cz = 0$.

Exercise 1 Find the point on the lines $[2, -1, 3]$ and $[4, -2, 5]$.

Example 1 We can think of an ordinary Euclidean point (x, y) as the projective point $(x, y, 1) = (\lambda x, \lambda y, \lambda)$. Then "ideal points" have 0 for their third coordinate. For example, the points on the line $y = x$ (or $[1, -1, 0] = [\lambda, -\lambda, 0]$) are of the form (x, x, z). Note that the ideal point $(x, x, 0)$ is not only on $[1, -1, 0]$, but also on all the Euclidean lines parallel to it, which are of the form $[1, -1, c]$, or more familiarly, $y = x + c$. However, there is no reason in projective geometry to single out any line or point as "ideal" or different from any other; homogeneous coordinates make all points and lines equivalent. ●

Example 2 We can relate each point of the projective plane to two opposite points on a sphere. For each projective point (x, y, z), there are two scalars λ and $-\lambda$ such that $(\lambda x, \lambda y, \lambda z)$ and $(-\lambda x, -\lambda y, -\lambda z)$ have length 1 and so are on the sphere. Projective lines correspond to great circles. Example 3 explains why the great circle corresponding to $[a, b, c]$ is in the plane perpendicular to the point (a, b, c). The identification of opposite points on the sphere gives single elliptic geometry. (See Section 3.5.) The points and lines of single elliptic geometry match exactly the points and lines of projective geometry. Single elliptic geometry has distance and angle measure in addition to the projective notions. ●

Example 3 Linear algebra in $\mathbf{R}^3$ matches projective geometry. A projective point is a one-dimensional subspace (a line through the origin). The scalar λ merely moves a point along this line. A projective line is a two-dimensional subspace (a plane through the origin). Recall that $[a, b, c]$ contains points (x, y, z), with $ax + by + cz = 0$ or $[a, b, c] \cdot (x, y, z) = 0$. The dot product of two vectors is 0 whenever the vectors are perpendicular. So $[a, b, c]$ must be the Euclidean plane through the origin perpendicular to the vector (a, b, c). Linear transformations take subspaces to subspaces, so projective transformations have a ready representation. ●

Exercise 2 Verify that the interpretation of Example 3 satisfies Axioms (i), (ii), and (iii) of a projective plane from Section 6.2.

Example 4 Three distinct points (p, q, r), (s, t, u), and (v, w, x) are collinear iff the determinant $\begin{vmatrix} p & s & v \\ q & t & w \\ r & u & x \end{vmatrix}$ is 0.

Solution. The three points are on the line $[a, b, c]$ iff $[\,a \quad b \quad c\,]\begin{bmatrix} p & s & v \\ q & t & w \\ r & u & x \end{bmatrix} = \vec{0}$, which gives a system of three homogeneous equations. From linear algebra, this system has a nonzero solution $[a, b, c]$ (that is, the points are collinear) iff the matrix is singular and the determinant is 0. ●

Representing projectivities, harmonic sets, and separation analytically is easier if we use two homogeneous coordinates for collinear points. We represent the points on a line as (u, v), where $(u, v) = (\lambda u, \lambda v)$, for $\lambda \neq 0$. We postpone the treatment of projectivities to Section 6.4 because they are a type of transformation.

Example 5 The points on $[1, -1, 2]$ are of the form (x, y, z), where $x - y + 2z = 0$ or $y = x + 2z$. This last equation enables us to eliminate the y-value, so the two coordinates (x, z) are sufficient to describe which point on this line we are considering. From the Euclidean point of view, the point $(x, 1) = (\lambda x, \lambda)$ corresponds to the real number x and $(1, 0) = (\lambda, 0)$ corresponds to ∞ or the ideal point. By solving the equation $y = x + 2z$ for x or z, we could eliminate either of these variables instead. These methods of determining two homogeneous coordinates (and others) are compatible; they correspond in linear algebra to changes of coordinates. ●

6.3.1 Cross ratios

The concept of the cross ratio of four collinear points, initially explored by the ancient Greeks for Euclidean geometry, provides the analytic key for both harmonic sets and separation. We use the notationally easy form of the cross ratio given in Example 6 whenever possible. Theorem 6.3.1 shows that our analytic interpretation matches Axiom (x), the most powerful separation axiom.

Example 6 For subscripts as in Axiom (x), the cross ratio of four distinct collinear points X_a, X_b, X_c, and X_d is $R(a, b, c, d) = \frac{a-c}{a-d} \div \frac{b-c}{b-d}$. Recall that $H(0\,1, x\frac{x}{2x-1})$. For example, $R(0, 1, 3, \frac{3}{5}) = \frac{-3}{-3/5} \div \frac{-2}{2/5} = -1$. Verify that for any $x \neq 0$, $\frac{1}{2}$, or 1 we have $R(0, 1, x, \frac{x}{2x-1}) = -1$. ●

Definition 6.3.1 The *cross ratio* of four collinear points $P = (p, q)$, $S = (s, t)$, $U = (u, v)$, and $W = (w, x)$ is

$$R(P, S, U, W) = \frac{\begin{vmatrix} p & u \\ q & v \end{vmatrix}}{\begin{vmatrix} p & w \\ q & x \end{vmatrix}} \div \frac{\begin{vmatrix} s & u \\ t & v \end{vmatrix}}{\begin{vmatrix} s & w \\ t & x \end{vmatrix}}.$$

Remark If $q = t = v = x = 1$, this formula reduces to the special case of Example 6.

Interpretation For four collinear points P, S, U, and W, $H(PS, UW)$ iff $R(P, S, U, W) = -1$ and $PS//UW$ iff $R(P, S, U, W) < 0$. Cross ratios, harmonic sets, and separation for lines are given dually.

Exercise 3 Suppose that $a < c < b < d$. Use Example 6 to verify that $R(a, b, c, d) < 0$ and so $X_aX_b//X_cX_d$.

Theorem 6.3.1 Axiom (x) holds in the analytic projective plane.

Proof. Let $a < b < c$ be real numbers. Then the homogeneous coordinates of the four points of the axiom are $X_a = (a, 1)$, $X_b = (b, 1)$, $X_c = (c, 1)$, and $X = (1, 0)$. Then

$$R(X_a, X_c, X_b, X) = \frac{\begin{vmatrix} a & b \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix}} \div \frac{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & 1 \\ 1 & 0 \end{vmatrix}} = ((a-b)/-1) \div ((c-b)/-1) = (a-b)/(c-b).$$

As $a < b < c$, this fraction is negative and so $X_aX_c//X_bX$, as required. ■

6.3.2 Conics

Projective geometry provides a unified way to study conics. The general equation of a conic in usual coordinates is $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$. In homogeneous coordinates this equation becomes $ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 = 0$. Every term is now of second degree in the variables x, y, and z, so we say that it is a *homogeneous* second-degree equation. (Homogeneous first-degree equations, $ax + by + cz = 0$, represent lines.) Because a conic has a second-degree equation, it can intersect a line in zero, one, or two points. Linear algebra makes the homogeneous equation even more useful, as Exercise 4 illustrates. The matrix form explains why we choose to have the factors of 2 in the general equation. However, not every such matrix gives a conic, as Exercise 5 illustrates. Quadratic forms in linear algebra are closely tied to conics and their generalizations. (See Fraleigh and Beauregard [3, Chapter 7].)

Exercise 4 Show that a point $P = (x, y, z)$ is on the conic with equation $ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 = 0$ iff $P^TCP = 0$, where P^T is the transpose of P and $C = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}$.

Exercise 5 Verify that $C = \begin{bmatrix} 2 & -3/2 & 1 \\ -3/2 & 1 & -1/2 \\ 1 & -1/2 & 0 \end{bmatrix}$ has a determinant of 0 and represents the product of the two lines $2x - y = 0$ and $x - y + z = 0$.

Interpretation By a *conic* we mean a symmetric invertible 3×3 matrix. Two such matrices represent the same conic iff one is the multiple of the other by some real number $\lambda \neq 0$. A point P is on a conic C iff $P^TCP = 0$.

From a projective viewpoint, circles, ellipses, parabolas, and hyperbolas—the Euclidean types of conics—are indistinguishable. However, if, as in Fig. 6.11, we arbitrarily designate one line k as an ideal line, hyperbolas intersect the ideal line in two points, parabolas intersect it in one point, and circles and ellipses do not intersect it. (To distinguish circles from ellipses requires measures of angles or distances.) We define a *tangent* to a conic to be a line with only one point on the conic. (In Euclidean geometry this definition fails, for the parabola $y = x^2$ and the hyperbola $y = \frac{1}{x}$ each have only one point of intersection with vertical lines $x = b$, but vertical lines aren't tangents. For the hyperbola we require $b \neq 0$.) The asymptotes to a hyperbola in Euclidean geometry are simply tangents in projective geometry.

Exercise 6 Verify $x^2 - yz = 0$ and $xy - z^2 = 0$ are the homogeneous equations for the conics $y = x^2$ and $y = \frac{1}{x}$. Explain why in projective geometry the (Euclidean vertical) lines $[1, 0, b]$ are not tangent to these conics. (Assume that $b \neq 0$ for the second conic.)

Two distinct Euclidean circles intersect in at most two points, but Exercise 7 below shows that two conics can intersect in more than two points. Five points, no three collinear, are required to completely determine a conic.

Exercise 7 Graph $y = x^2$ and $x^2 + (y-2)^2 = 2$ and find their intersections.

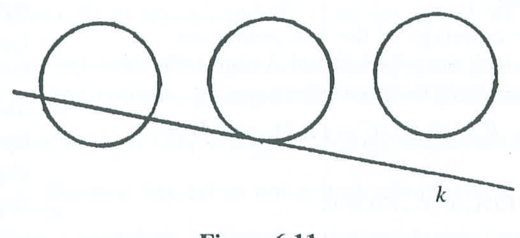

Figure 6.11

PROBLEMS FOR SECTION 6.3

1. a) Find the point of intersection of the lines $[m, -1, b]$ and $[m, -1, b']$. Interpret this situation in Euclidean geometry.
   b) Verify that the line $[a, b, c]$ on the points (p, q, r) and (s, t, u) is given by (the transpose of) their cross product, or equivalently by $\left[\begin{vmatrix} q & t \\ r & u \end{vmatrix}, \begin{vmatrix} p & s \\ r & u \end{vmatrix}, \begin{vmatrix} p & s \\ q & t \end{vmatrix}\right]$, where $\begin{vmatrix} v & y \\ w & z \end{vmatrix}$ is the determinant of a 2×2 matrix. Describe how you can find the intersection of two lines.
2. Follow Van Staudt's coordinatization of Section 6.2, using $O = (0, 0, 1)$, $X = (1, 0, 0)$, $Y = (0, 1, 0)$, and $U = (1, 1, 1)$. Find homogeneous coordinates for X_a, Y_b, and P_{ab}. Find homogeneous coordinates for the point of intersection of $[0, 0, 1]$ and $\overleftrightarrow{OP_{ab}}$.
3. a) Verify that no three of points $(0, 0, 1)$, $(6, 0, 1)$, $(3, 3, 1)$, and $(8, 4, 1)$ are collinear. Find the diagonal points of the complete quadrangle they form and verify that these diagonal points are not collinear. (See Problem 7 of Section 6.2.) Use the interpretation of Example 1 to graph the complete quadrangle and its diagonal points.
   b) Pick two of the diagonal points of part (a), say, A and B, and find the coordinates of the line k on them. Find the intersections C and D of k with the other two sides of the complete quadrangle. Explain why the four points A, B, C, and D form a harmonic set. Verify that they form a harmonic set by using a cross ratio.
4. Use Example 6 to verify Axioms (v), (vi), and (vii) and Theorem 6.2.1, part (iii).
5. a) Illustrate Axiom (viii) by using five points on a horizontal line. If the coordinates of the points, from left to right, are a, b, c, d, and e, how do these letters match the letters of the points in the axiom? Use the form of the cross ratio in Example 6 to verify Axiom (viii) for the case in your drawing.
   b) What other orderings of the five points are compatible with the hypothesis of Axiom (viii)? Verify Axiom (viii) for these other cases.
6. Let $A = (1, 0)$, $B = (0, 1)$, $C = (1, 1)$, and $D = (r, 1)$.
   a) Verify that $R(A, B, C, D) = r$.
   b) There are 24 orderings of the four points A, B, C, and D, but there are only six different cross ratios. Use the coordinates in part (a) to find the five other values of the cross ratios besides r. List the orderings that have the same cross ratio, r, as the ordering A, B, C, D.
   c) Explore whether, for any four distinct collinear points, the relationship of the six different cross ratios of part (b) holds.
   d) Explore what values the cross ratio $R(A, B, C, D)$ can have for three distinct points, with one of them repeated.
7. a) In ordinary analytic geometry $y = x/(x + 1)$ is a hyperbola with asymptotes $y = 1$ and $x = -1$. Convert these equations to homogeneous equations. Find the ideal point of each asymptote and verify that it is the intersection of the asymptote and the conic.
   b) Find the asymptotes of $x^2 - y^2 = 1$ and repeat part (a) for this conic.
   c) Convert the equation of the parabola $y = x^2 - x$ to homogeneous form and verify that the ideal line $[0, 0, 1]$ has one point of intersection with it.
8. Explain why a general point on $x^2 + y^2 - z^2 = 0$ (the unit circle) is of the form $(\cos\alpha, \sin\alpha, 1)$. (In particular, why can you always choose $z = 1$?) Use analytic geometry to find the homogeneous coordinates of the line tangent to this circle at this point. [*Hint:* The coordinates of these tangents and their points of tangency have a remarkable property. (The set of tangents forms a *line conic*, the dual of a point conic, which is called a conic.)]
9. a) Show the general equation for the family of conics through the four points $(1, 1, 1)$, $(1, -1, 1)$, $(-1, 1, 1)$, and $(-1, -1, 1)$ to be $ax^2 + cy^2 + fz^2 = 0$, where $a + c + f = 0$.
   b) Use Example 1 to graph the three degenerate conics through these four points and find their equations. (Each is a pair of lines.)
   c) Let $f = 1$ and pick specific values for a and c. Graph the resulting conic.
   d) Show every point (p, q, r) to be on just one of the conics in the family of part (a), including the degenerate ones of part (b).
   e) What types of Euclidean conics are in the family of part (a)?

10. (Calculus) a) Write the equation $y = x^2$ in homogeneous coordinates.
    b) Show that the lines $[a, b, c]$ tangent to the equation in part (a) satisfy the equation of the line conic $a^2 - 4bc = 0$ as follows. First, pick x_0, a real number, and find the coordinates of the point $(x_0, y_0, 1)$ on the conic. Next use calculus to find the equation of the tangent line. Now convert this line to the form $[a, b, c]$ and verify that it satisfies the equation $a^2 - 4bc = 0$.
    c) Graph $y = x^2$ and selected tangents.
11. (Calculus) Repeat Problem 10 for the hyperbola $y = \frac{1}{x}$ and the corresponding line conic $4ab - c^2$. Explain the similarities in these problems from a projective viewpoint.
12. Suppose that P is a point on the conic C. Claim: The line $l = P^T C$ is tangent to C at P.
    a) Verify the claim by using the matrix form for conics in Problems 10 and 11.
    b) Show that P is on the line l of the claim.
    c) Suppose that P and Q are two points on the conic C and that l and k are their corresponding lines in the claim. Show that $l = k$ iff $P = Q$. Explain why this equation shows that the lines in the claim intersect the conic in only one point. Write a proof of the claim.
    d) Show that all tangents l satisfy the equation of the line conic $lC^{-1}l^T = 0$.