6.2 Axiomatic Projective Geometry

In this section we develop axiomatically some elementary properties of projective planes. (For a more thorough axiomatic development, see Coxeter [2] or Tuller [9].) The first five axioms describe the minimum relations of points, lines and harmonic sets. (Recall that we defined harmonic sets in Section 6.1 in terms of points and lines.) The separation axioms, which appear later in this section, provide a structure of the points on a line analogous to the notion of order. The undefined terms are *point*, *line*, *on*, and *separate*.

Axioms 6.2.1

i) Two distinct points have exactly one line on them.

ii) There are at least four points with no three on the same line.

iii) Every two distinct lines have at least one point on both lines.

iv) Given three distinct points P, Q, and R on a line k, there is a unique point S on k, distinct from P, Q, and R, such that $H(PQ, RS)$.

v) If $H(PQ, RS)$, then $H(RS, PQ)$.

Theorem 6.2.1

i) Two distinct lines have exactly one point on both lines.

ii) Every line has at least four distinct points on it.

iii) If $H(PQ, RS)$, then $H(PQ, SR)$.

Proof. See Problem 1 for parts (i) and (iii). For part (ii), let k be any line. By Axiom (ii) there are four points A, A_1, A_2, and A_3 and at least one of them, say, A, is not on k. Consider the lines k_i on A and A_i, for $i = 1, 2, 3$. By Axiom (ii) these lines are distinct. By part (i) these lines each have one point in common with k, which gives three distinct points on k. Axiom (iv) guarantees the fourth point. ■

Definition 6.2.1 The point on the lines k and l is denoted $k \cdot l$. Points on the same line are *collinear*, and lines on the same point are *concurrent*.

We follow Karl Van Staudt's method of using harmonic sets to construct and coordinatize infinitely many points on a line and the plane without any dependence on distance. The subscripts reflect Problem 9 (b) of Section 6.1, which showed that two given points, their Euclidean midpoint, and the ideal point formed a harmonic set.

Definition 6.2.2 Given three distinct collinear points X_0, X_1, and X, define X_2 to be the point such that $H(XX_1, X_0X_2)$. Given X_n and X_{n+1}, define X_{n+2} to be the point such that $H(XX_{n+1}, X_nX_{n+2})$. Given X_a and X_b, define $X_{(a+b)/2}$ to be the point such that $H(XX_{(a+b)/2}, X_aX_b)$ (Fig. 6.6).

Exercise 1 Define points X_{-n}, for n a positive integer.

Example 1 Figure 6.6 illustrates the placement of various points X_a. If X is the ideal point on the horizon of a perspective painting, this construction shows how to place points so that they look equally spaced. By Problem 4, the sequence of Euclidean distances $d(X_n, X)$ forms a harmonic sequence, such as $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ ●

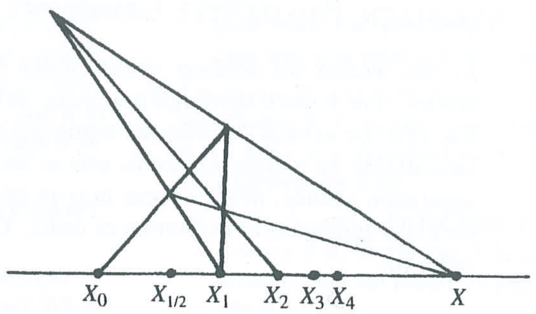

Figure 6.6

The first five axioms don't guarantee that all the points X_a from Definition 6.2.2 are distinct. (Indeed, in Chapter 7 we demonstrate that a finite projective plane with just four points on a line satisfies all five axioms.) Theorem 6.2.3 depends on the following separation axioms to ensure that the X_a are distinct. We write $PQ//RS$ to denote that P and Q separate R and S. Separation in projective geometry takes the role of Euclidean geometry's betweenness.

Axioms 6.2.2 Separation Axioms

vi) If $PQ//RS$, then P, Q, R, and S are distinct collinear points, $PQ//SR$ and $RS//PQ$.

vii) If P, Q, R, and S are distinct collinear points, then at least one of the following holds: $PQ//RS$, $PR//QS$, or $PS//QR$.

viii) If $PQ//RS$ and $PR//QT$, then $PQ//ST$.

ix) If $H(PQ, RS)$, then $PQ//RS$.

Theorem 6.2.2 **i)** If $PQ//RS$, then $QP//RS$, $QP//SR$, $RS//QP$, $SR//PQ$, and $SR//QP$.

ii) If A, B, C, and D are distinct collinear points, then exactly one of the following holds: $AB//CD$, $AC//BD$, or $AD//BC$.

Proof. See Problem 3. ■

Theorem 6.2.3 If X_p and X_q are determined from Definition 6.2.2 and Exercise 1 and $p \neq q$, then X_p and X_q are distinct.

Partial Proof. Show that, if $a < b < c$ for nonnegative integers (and so X_a, X_b, and X_c are determined from the definition), then $XX_b//X_aX_c$. Axiom (vi) then forces these points to be distinct. The remaining cases are similar.

Use induction to show that $XX_b//X_aX_c$, where $0 \leq a < b < c \leq k+1$ holds for all integers $k \geq 1$. We have $H(XX_1, X_0X_2)$, so Axiom (ix) gives the case $k = 1$. Now assume that the case $0 \leq a < b < c \leq k+1$ holds and let $0 \leq a < b < c \leq k+2$. If $c < k+2$, we are back at the induction hypothesis. Suppose that $c = k+2$. Then, by definition, $H(XX_{k+1}, X_kX_{k+2})$. Axiom (ix) gives $XX_{k+1}//X_kX_{k+2}$. If $a = k$, then $b = k+1$ and we are done. If $0 \leq a < k$, we have $XX_k//X_{k+1}X_a$ by the induction

JEAN VICTOR PONCELET

Jean Victor Poncelet (1788–1867) grew up in France during revolutionary times. He studied at the Ecole Polytechique under the influence of the legendary Monge. He then became an officer in Napoleon's army in the ill-fated campaign against Russia. He spent a year during 1813 and 1814 in a Russian prison. In prison he had the opportunity to reflect and write. He reconstructed his geometric education from memory and went on to discover many new results. Poncelet worked for various French governments after the fall of Napoleon and occasionally taught.

Projective geometry became a separate subject and moved to prominence with publication in 1822 of Poncelet's treatise on the subject, a revised and expanded version of his prison musings. Poncelet was an outspoken advocate of the synthetic approach, following the dictum of Lazare Carnot: ". . . to free geometry from the hieroglyphics of analysis." He realized the power of the general analytic approach compared to the isolated proofs of classical geometry. However, he felt that analytic geometry gave answers without giving insight. He developed projective geometry to provide general methods within the synthetic tradition. In the process he rediscovered many of the properties found previously by Desargues and others but then forgotten. He looked for properties of figures preserved by perspectivities. He was the first to realize the importance of duality.

The principle of continuity, which he used implicitly as an axiom, provided a powerful method for generalizing many results from understood cases to analogous cases. Thus a shape can pass continuously from circles and ellipses to parabolas to hyperbolas, much as the shadow of a lamp shade on a wall does as the lamp is tilted. The projective properties of tangents and other objects transfer along with the changing curves. For example, an asymptote of a hyperbola becomes just a special type of tangent. He even extended the principle of continuity to explore imaginary intersections of lines and conics that don't intersect in real points. Thus Poncelet initiated the study of complex projective geometry, although without using coordinates.

hypothesis. Axiom (viii), with $P = X$, $Q = X_{k+1}$, $R = X_k$, $S = X_{k+2}$, and $T = X_a$, gives $XX_{k+1}//X_{k+2}X_a$, which Theorem 6.2.2 converts to $XX_{k+1}//X_aX_{k+2}$. If $b = k+1$, we are done. Problem 8 considers the remaining case, $a < b < k+1$. ■

The final axiom, the continuity axiom, extends the strategy of Theorem 6.2.3 to ensure that a projective line includes all points X_r, where r is a real number, together with the additional point X. Figure 6.7 illustrates how to match all but one of the points on a circle with the points on a line. In effect the final axiom ensures that the points on a projective line are arranged like the points on a circle. Visualizing lines in perspective drawings as circles is difficult because movement can be along the line in two directions, but only one seems to go toward the horizon. However, railroad tracks appear to intersect at the horizon in both directions. Theorem 6.2.1 forces each line to intersect the horizon (ideal line) in just one point, so these two directions must somehow meet at the same ideal point on the horizon, completing a circle. The (topological) arrangement of points of the entire projective plane is the same as the points in single elliptic geometry. (See Section 3.5.)

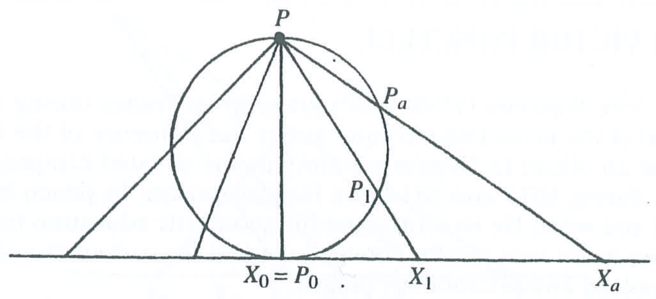

Figure 6.7

Axiom 6.2.3 **Continuity Axiom** Given any three points X, X_0 and X_1 on a projective line, there is a one-to-one correspondence between the real numbers r and all the points X_r on the line except X such that b is between a and c iff $X_aX_c//X_bX$.

Section 6.1 described the projective plane in terms of the familiar Euclidean plane together with an added "horizon," a line of ideal points. Based on the axioms presented we can now confirm that description. From Axiom (ii) we can start with four points O, X, Y, and U, no three of which are collinear. Think of $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$ in Fig. 6.8 as the x- and y-axes and O as the origin for the points of the Euclidean plane. Axiom (x) and the discussion preceding Theorem 6.2.2 enable us to fill out the lines $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$, provided we can find three points on each. For $\overleftrightarrow{OX}$ we use $O = X_0$, $\overleftrightarrow{OX} \cdot \overleftrightarrow{YU} = X_1$ and X. Similarly, for $\overleftrightarrow{OY}$ we use $O = Y_o$, $\overleftrightarrow{OY} \cdot \overleftrightarrow{XU} = Y_1$ and Y. Thus we have all the points X_a and Y_b on the "axes" $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$. Next we include the points P_{ab} corresponding to the Euclidean plane, using two numbers, just as coordinates in analytic geometry. In Fig. 6.8, P_{ab} is the intersection of $\overleftrightarrow{X_aY}$ and $\overleftrightarrow{XY_b}$. For any point P not on the line $\overleftrightarrow{XY}$, $\overleftrightarrow{PY}$ intersects $\overleftrightarrow{OX}$ at some point X_a and $\overleftrightarrow{PX}$ intersects $\overleftrightarrow{OY}$ at some point Y_b, which gives $P = P_{ab}$. Thus the points not on line $\overleftrightarrow{XY}$ look like the Euclidean plane. In effect, $\overleftrightarrow{XY}$ is the line of ideal points. However, the points on $\overleftrightarrow{XY}$ don't have natural coordinates in this procedure. Indeed, we used up all possible pairs (a, b) to label the points P_{ab} not on

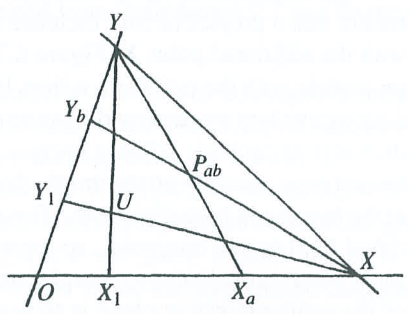

Figure 6.8

$\overleftrightarrow{XY}$. In Section 6.3 we present homogeneous coordinates, which are an elegant solution to this problem, involving the use of three coordinates for each point.

6.2.1 Duality

Poncelet noticed that, without parallel lines, lines have the same properties as points in projective plane geometry. That is, the words *point* and *line* in any axiom or theorem of projective geometry (and any definitions used in this statement) can be exchanged to get another theorem, called the *dual*. For example, the dual of "Every line has infinitely many points on it" is "Every point has infinitely many lines on it." Similarly, *collinear* and *concurrent* are dual concepts, as are *complete quadrangles* and *quadrilaterals*. People think about points differently from lines, so this structural similarity was (and is) hard to see. However, duality is an aesthetically pleasing property, which doubles the number of theorems available, often giving us theorems we might not have imagined.

Exercise 2 State the duals of Axioms (i), (ii), and (iii).

Theorem 6.2.4 The duals of Axioms (i), (ii), and (iii) hold.

Proof. See Problem 6. ■

Next we develop the duals of harmonic sets of points and separation of points. The labeling of Fig. 6.9 provides the key to connecting a harmonic set of points $H(PQ, RS)$ and a harmonic set of lines $H(pq, rs)$. In particular, a harmonic set of points $H(PQ, RS)$ and a point A not on their line determine a harmonic set of the lines through A and these points. Conversely, a harmonic set of lines $H(pq, rs)$ and a line not on their common point determine a harmonic set of points. Similarly, Definition 6.2.3 uses the separation of points to define the separation of lines. (We assume that the definition is well-defined; that is, different choices of the line k give the same result of whether or not $pq//rs$ or $H(pq, rs)$. See Tuller [9] for more information.)

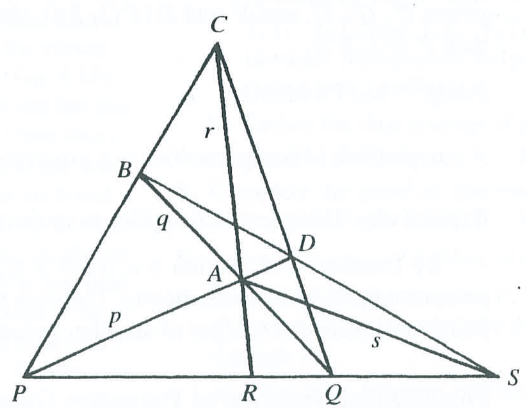

Figure 6.9

Definition 6.2.3 Let p, q, r, and s be concurrent lines on O, k be any line not on O, and $P = p \cdot k$, $Q = q \cdot k$, $R = r \cdot k$, and $S = s \cdot k$. Define $pq//rs$ iff $PQ//RS$ and $H(pq, rs)$ iff $H(PQ, RS)$.

Theorem 6.2.5 The duals of Axioms (iv)–(x) hold.

Proof. We prove the dual of Axiom (iv). (See Problem 6 for the others, which are similar.)

The dual of Axiom (iv) states, "Given three distinct lines p, q, and r on a point O, there is a unique line s on O, distinct from p, q, and r, such that $H(pq, rs)$." To prove this axiom, we start with the three lines p, q, and r on O. Let k be any line not on O. By Axiom (iii), k intersects the lines p, q, and r in the points P, Q, and R, respectively. By Theorem 6.2.1, P, Q, and R are distinct. Hence the hypothesis of Axiom (iv) holds, and there is a unique point S on k distinct from P, Q, and R such that $H(PQ, RS)$. By Theorem 6.2.1 $s = \overleftrightarrow{OS}$ is distinct from p, q, and r. Definition 6.2.3 gives $H(pq, rs)$. Finally, note that, as S is unique, s also is unique. ■

Once we have the duals of the axioms, the duals of all the theorems follow immediately. Indeed, we could mechanically write the proof of a dual by switching the words *point* and *line* and so on throughout the original proof.

Exercise 3 Write the duals of Theorems 6.2.1, 6.2.2, and 6.2.3.

6.2.2 Perspectivities and projectivities

The notion of a perspectivity originates in perspective drawing. Recall that a perspectivity maps the points P_i of one line to the points Q_i of another line, using a point O such that for each i, O, P_i, and Q_i are collinear. Theorem 6.2.6 shows that harmonic sets and the relation of separation are preserved in perspectivities and so in perspective drawings.

Theorem 6.2.6 A perspectivity preserves harmonic sets of points and the relation of separation. That is, if a perspectivity from O maps the collinear points P, Q, R, and S to the collinear points P', Q', R', and S' and $H(PQ, RS)$, then $H(P'Q', R'S')$. Similarly, if $PQ//RS$, then $P'Q'//R'S'$.

Proof. See Problem 9. ■

Definition 6.2.4 A composition of perspectivities is a *projectivity*.

Exercise 4 Explain why Theorem 6.2.6 applies to projectivities as well as perspectivities.

By Problem 2 of Section 6.1, there is a projectivity that maps any three collinear points to any three collinear points. Theorem 6.2.7 reveals that the images of these three points determine the images of all other points on the original line.

Theorem 6.2.7 **Fundamental Theorem of Projective Geometry** A projectivity of a line is completely determined by three points on the original line and their images.

Proof. WLOG call the three points on the original line X, X_0, and X_1 and suppose that the projectivity takes them to Y, Y_0, and Y_1, respectively. Theorem 6.2.5 ensures that any point X_p described in Theorem 6.2.3 must go to Y_p. Problem 2 and the continuity axiom extend this process, matching each X_r with Y_r. ■

Exercise 5 Define and illustrate the concept of a line perspectivity taking concurrent lines to concurrent lines. Write the duals of Theorems 6.2.6 and 6.2.7.

PROBLEMS FOR SECTION 6.2

1. **a)** Prove the rest of Theorem 6.2.1.

b) What other arrangements of P, Q, R, and S form harmonic sets for $H(PQ, RS)$? Prove your answer.

2. Describe all numbers q such that an X_q is constructed according to Definition 6.2.2. Explain why all positive real numbers r can be written as limits of these q.

3. Prove Theorem 6.2.2. [*Hint:* For part (ii) use Axiom (viii).]

4. Let r represent the Euclidean point $(r, 0)$.

a) In Problem 8 of Section 6.1, you saw that $H(0\ 1, a\ \frac{a}{2a-1})$. Use similarity to show, for $k > 0$, $H(0\ k, ak\ \frac{ak}{2a-1})$.

b) Show that $H(0\ \frac{1}{n+1}, \frac{1}{n}\ \frac{1}{n+2})$.

c) Use part (b) to show how to construct poles that appear equally spaced in a perspective painting.

5. Art books give various methods of constructing equally spaced poles in a perspective painting. (See Powell [6].) Explain how the following construction blends harmonic sets and Euclidean ideas. Start with the horizon line, the pole nearest the viewer, and the base of the next pole drawn in (Fig. 6.10). Extend line b connecting the two bases to the horizon line to find the appropriate ideal point. Draw line t connecting this ideal point with the top of the first pole. All the poles will have their bases on b and tops on t. Draw the second pole parallel to the first pole. Draw line p through the ideal point parallel to the poles. Draw line k_1 from the top of the first pole through the base of the second pole to point P on p. Line k_2 connecting the top of the second pole with P intersects b at the base of the third pole. Continue as in Fig. 6.10 to determine the other poles.

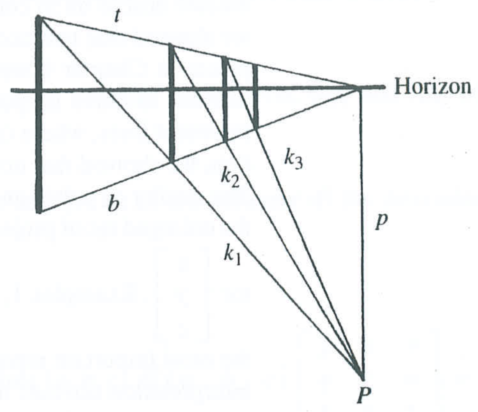

Figure 6.10

6. **a)** Prove Theorem 6.2.4.

b) State the duals of Axioms (v)–(x).

c) Prove the duals of Axioms (v)–(x).

7. **a)** In a complete quadrangle with vertices T_1, T_2, T_3, and T_4, the *diagonal points* are the three points of intersection of the "opposite" sides: $\overleftrightarrow{T_1T_2} \cdot \overleftrightarrow{T_3T_4}$, $\overleftrightarrow{T_1T_3} \cdot \overleftrightarrow{T_2T_4}$, and $\overleftrightarrow{T_1T_4} \cdot \overleftrightarrow{T_2T_3}$. Draw a picture to illustrate this situation and prove that the diagonal points aren't collinear.

b) Define the dual concept of part (a) and illustrate the dual theorem.

8. Complete the proof of Theorem 6.2.3. [*Hint:* Use Axiom (viii) with $P = X$.]

9. Prove Theorem 6.2.6 [*Hint:* $H(\overleftrightarrow{OP}\ \overleftrightarrow{OQ}, \overleftrightarrow{OR}\ \overleftrightarrow{OS})$.]

10. For a fixed Euclidean point P, let k_i be the line on P having slope i and let k be the vertical line through P.

a) Does this labeling of lines correspond to the labeling of points on a projective line?

In particular, do the duals of the separation axioms hold?

b) In analytic geometry, if m_1 and m_2 are the bisectors of the angles formed by m_3 and m_4, then $H(m_1m_2, m_3m_4)$. Investigate whether this relation holds for the labeling of the lines given, following the construction of a harmonic set of lines from Problem 7 of Section 6.1, with $m_1 = k$ and $m_2 = k_0$.