6

Projective Geometry

The artists of the Renaissance uncovered the rules of perspective. Projective geometry provides the foundation for perspective both in art and computer graphics. We explore these aspects of projective geometry and more.

> Metrical geometry is thus a part of [projective] geometry, and [projective] geometry is all geometry. —*Arthur Cayley*

6.1 Overview and History

Projective geometry grew out of the study of perspective in Renaissance art. Albrecht Dürer (1471–1528), Leonardo da Vinci (1452–1519) and other artists worked out the geometric rules of perspective. These rules weren't particularly complicated, but they transformed the appearance of paintings. One key insight clearly contradicting Euclidean geometry was that formerly parallel lines met at "ideal points" on the horizon, an ideal line intuitively infinitely far away. However, these painters were *not* challenging Euclidean geometry. (Indeed Euclid had written about such effects of perception in his work, *The Optics*.) The attempt to paint the way people saw the world emerged from a philosophy that took people as "the measure of all things," rather than viewing the world from a divine point of view.

Girard Desargues (1593–1662) was the first person to prove geometric properties beyond what was needed for painting. Unfortunately, his notes were quite hard to read and had unusual conventions, such as using flowers and trees as the names of geometric objects. Only a few mathematicians, including Blaise Pascal (1623–1662), built on Desargues's efforts and saw the unifying power of what we now call projective methods. Johannes Kepler (1571–1630), who used conics effectively in his astronomical work, realized that circles stretch continuously to ellipses and then to parabolas and hyperbolas. The shadow of a lamp shade cast on a wall illustrates this process, suggesting the notion of projective transformations.

However, the progress made by these pioneers was overshadowed and mostly forgotten because of the marvelous advances of analytic geometry and then calculus, which quickly dominated mathematical thought. Led by Gaspard Monge, the reconsideration and rediscovery of projective ideas started after 1800. One of his students, Jean Victor Poncelet (1788–1867) developed this subject extensively and accelerated others' research with his book, published in 1822, emphasizing a synthetic approach to geometry that featured properties that didn't depend on distance. Poncelet realized the importance of duality, which says that points and lines function the same way. At that time analytic methods seemed useless in this geometry because projective geometry added "ideal points," whereby parallel lines meet on the horizon, and these points had no obvious coordinates. The success of the projective approach launched a rivalry between geometers who advocated an analytic approach and those, including Poncelet, who pursued projective geometry synthetically.

Augustus Möbius and Julius Plücker developed coordinates for projective geometry, and gradually geometers realized that the synthetic and analytic approaches complemented each other. Möbius initiated the study of transformations in projective geometry and other geometries. Karl Van Staudt (1798–1867) showed how to derive projective coordinates independent of Euclidean geometry. Arthur Cayley (1821–1895) and Felix Klein showed how to develop Euclidean, hyperbolic, and single elliptic geometries within projective geometry. Later mathematicians showed that the geometry of the spe-

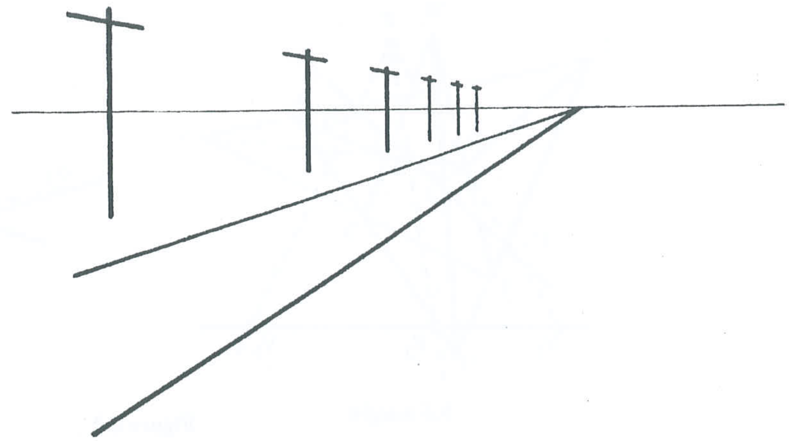

Figure 6.1

..al theory of relativity is also a part of projective geometry. Klein realized the vital role projective transformations played in unifying geometry. In the twentieth century mathematicians have regularly used analytic projective geometry in various fields, including computer-aided design, algebraic geometry, and statistical design theory.

6.1.1 Projective intuitions

Perspective painting depends on the idea of projecting a scene onto the plane of the easel with respect to the eye of the painter. Figure 6.1 illustrates such a projection, showing parallel lines that appear to meet at the horizon. Axiomatically we characterize projective geometry by the axiom that every two distinct lines in the plane have at least one point on both lines. The definition of a perspectivity captures the idea of perspective drawing.

Definition 6.1.1 A *perspectivity with respect to a point* T is a mapping of the points U_i on one line to the points V_i on another line so that V_i is the image of U_i iff U_i, V_i, and T are on the same line (Fig. 6.2).

In Euclidean geometry, the distance between two points is a fundamental measurement. As the spacing of the telephone poles in Fig. 6.1 and the drawing in Fig. 6.2(a) suggest, perspective distorts distances and proportions. Moreover, the ordering of points on a line can change, as indicated in the drawing in Fig. 6.2(b). That is, projective transformations can move points around so that one no longer needs to lie between the other two. (Artists never need to choose a perspective that switches the order.)

In Problem 2 you are asked to show that a pair of perspectivities can map any three points on one line to any three points on another line. That is, properties involving only two or three points on a line, such as distance or betweenness, are not projective properties. Similarly, angle measure is not a projective property. Geometers have found properties involving four or more points, built on straight-line constructions, and preserved by all projective transformations. The most important of these properties, a

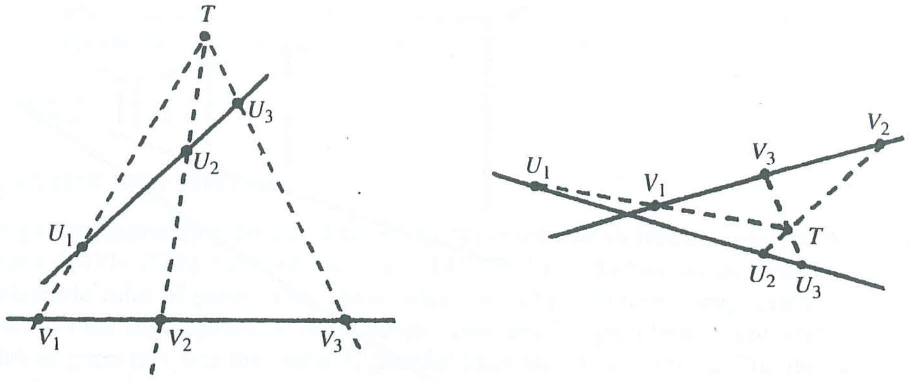

Figure 6.2

harmonic set, is built from a complete quadrangle: Four points, no three on the same line, such as T_1, T_2, T_3, and T_4 in Fig. 6.3, and the six lines they determine form a *complete quadrangle*. The points P, Q, R, and S form a harmonic set.

Definition 6.1.2 Four distinct points P, Q, R, and S on a line k form a *harmonic set*, denoted $H(PQ, RS)$, iff there is a complete quadrangle with six lines different from k such that P is on two of the lines, Q is on two other lines, R is on one other line, and S is on the remaining line.

Example 1 From three points P, Q, and R on a line, the following construction determines the unique fourth point S to form a harmonic set, as shown in Fig. 6.3. Although not obvious, every such construction will give the same fourth point. (See Problem 4.) Indeed, this uniqueness is an axiom in Section 6.2.

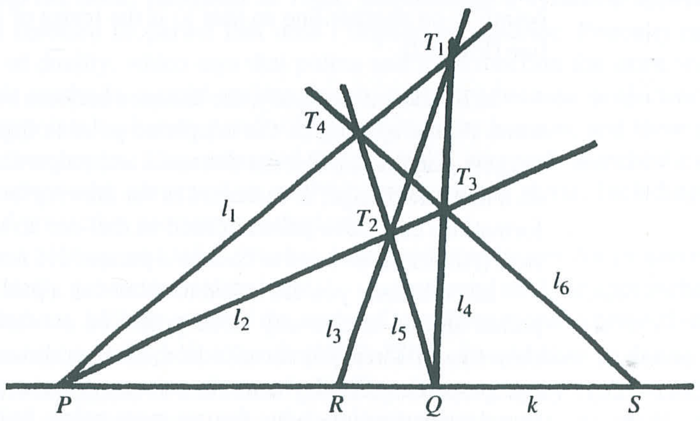

Figure 6.3

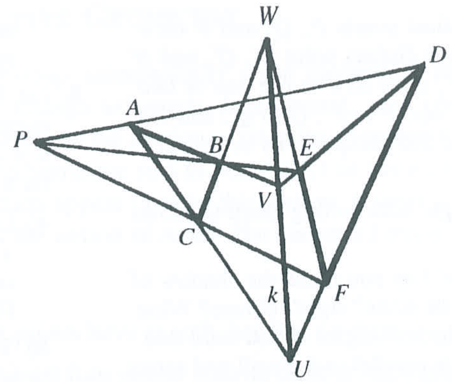

Figure 6.4

Let P, Q, and R be any three points on a line k. Draw two arbitrary lines l_1 and l_2 through P and another arbitrary line l_3 through R. (All the lines l_i are distinct and differ from k.) Let T_1 and T_2 be the intersections of l_3 with l_1 and l_2, respectively. Let l_4 and l_5 connect Q with T_1 and T_2, respectively. Then l_4 intersects l_2, say, at T_3, and l_5 intersects l_1, say, at T_4. Finally, draw the line connecting T_3 and T_4 and find its intersection S with the original line k. The complete quadrangle with points T_i and lines l_j satisfies the conditions of the definition, showing $H(PQ, RS)$. ●

Exercise 1 A *complete quadrilateral* is a set of four lines, no three on the same point, and the six points they determine. Draw a complete quadrilateral.

Example 2 Two triangles $\triangle ABC$ and $\triangle DEF$ are said to be *perspective from the point P* whenever P is on the three lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$. Figure 6.4 illustrates this situation, as well as the following closely related concept. Two triangles $\triangle ABC$ and $\triangle DEF$ are said to be *perspective from the line k* whenever the corresponding sides of the triangles intersect on k. That is, $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$, $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$, and $\overleftrightarrow{BC}$ and $\overleftrightarrow{EF}$ have their three points of intersections on k. Desargues used three-dimensional Euclidean properties to show Desargues's theorem: If two triangles are perspective from a point, then they are perspective from a line. (See Project 2.) Desargues's theorem is often taken as an axiom, and then properties of harmonic sets are proven from it. ●

(See Kline [4, Chapters 14 and 35] for more on the history of projective geometry.)

PROBLEMS FOR SECTION 6.1

1. Draw a picture similar to Fig. 6.1 wherein the telephone poles are spaced geometrically. For example, let the first pole be 4 in. from the ideal point on the horizon where the poles meet, and the succeeding poles be $3 = 4 \cdot \frac{3}{4}$, $2.25 = 4 \cdot (\frac{3}{4})^2, \ldots$ in. from the ideal point. Do the distances between poles look like they are getting smaller or larger as they go toward the ideal point? Try other distance relationships between the poles. What relationship describes the placement of equally spaced items drawn in perspective, as in Fig. 6.1?

2. a) For any two distinct points P and Q on a line k and any two distinct points P' and Q' on another line k', show how to find a perspectivity from some point T that takes P to P' and Q to Q'.

b) For any three distinct points P, Q, and R on a line k and any three distinct point P', Q', and R' on another line k', show how to use one or two perspectivities to take P to P', Q to Q', and R to R'. [*Hint:* Use the first perspectivity to take R to R'.]

3. Use a lamp with a light bulb casting sharp shadows on a wall.

a) Cut out a triangle. Can you make the shadow of any of its angles be acute? right? obtuse? What happens to the shadow angles if you hold one side of the triangle parallel to the wall and rotate the triangle about that side?

b) Describe the shapes that the shadow of a cut out circle can be.

4. a) Use a dynamic geometry construction program to model the construction of Example 1. Keep P and Q fixed and move R along line k. Note how S moves in response to R.

b) Keep P, Q, and R fixed but move the lines through them. Does S move?

c) Have the program measure the distances $d(P, Q)$, $d(P, R)$, and $d(P, S)$. Set $d(P, Q)$ at some convenient distance, such as 1 or 10, and investigate the relationship of $d(P, R)$ and $d(P, S)$.

5. Use graph paper to construct a harmonic set $H(PQ, RS)$, where P, Q, and R are on the x-axis at coordinates 0, 1, and 3. What is the coordinate of S? Choose different initial lines and construct S from the same three points. Do your two constructions yield (nearly) the same point?

6. a) Prove that, if $H(PQ, RS)$, then $H(PQ, SR)$, $H(QP, RS)$, and $H(QP, SR)$.

b) Suppose that $H(PQ, RS)$. Use constructions to explore which, if any, of $H(PR, QS)$, $H(PS, QR)$, and $H(RS, PQ)$ can be harmonic sets.

7. A harmonic set of lines $H(jk, lm)$ relates to a complete quadrilateral in the same way that a harmonic sets of points relates to a complete quadrangle. The roles of points and lines are simply interchanged.

a) Define a harmonic set of lines.

b) Convert Example 1 to a construction of a harmonic set of lines. Carry out this construction twice, once where l is a bisector of j and k and once where the lines seem unrelated. In the first case how is m related to the other lines?

8. Let P, Q, R, and S be points on the x-axis with coordinates 0, 1, a, and $h(a)$, respectively, with $a \neq 2$ and $a \neq \frac{1}{2}$. Assume that $H(PQ, RS)$ and look for a formula for $h(a)$.

a) Let l_1, l_2, and l_3 be the lines $x = 0$, $y = x$, and $y = -x + a$. Verify that P is on l_1 and l_2 and R is on l_3. Find the points T_1 and T_2, as denoted in Fig. 6.3.

b) Verify that l_4 and l_5 are $y = -ax + a$ and $y = ax/(a-2) - a/(a-2)$.

c) Verify that $T_3 = (\frac{a}{1+a}, \frac{a}{1+a})$ and $T_4 = (0, \frac{a}{2-a})$.

d) Find the equation of l_6. Verify that $S = (h(a), 0)$, where $h(a) = a/(2a-1)$.

9. a) In Problem 8, alter one of the lines (l_1, l_2, or l_3) to show that, when $a = 2$, $h(2) = \frac{2}{3}$.

b) In Problem 8, for $a = \frac{1}{2}$ find the equation of l_6. What happens to S in this case? How does this situation fit with the equation in Problem 8 (d)?

10. In Fig. 6.5 the circle has the equation $(x - \frac{1}{2})^2 + y^2 = 0$, with $P = (0, 0)$, $Q = (1, 0)$, $R = (a, 0)$, and $0 < a < 1$, $a \neq \frac{1}{2}$. Find the coordinates of S, where $\overleftrightarrow{SA}$ and $\overleftrightarrow{SB}$ are tangents to the circle and A and B have a for their x-coordinate. Note that Problem 8 shows that $H(PQ, RS)$.

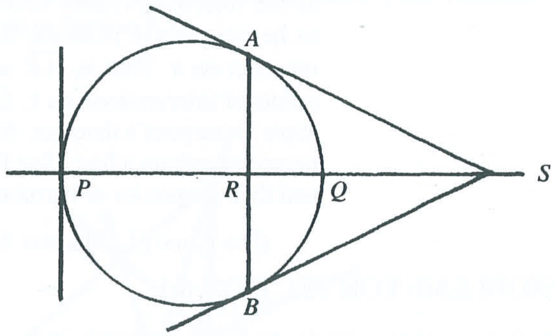

Figure 6.5

11. a) In Fig. 6.4 verify that $\triangle ABP$ and $\triangle UWF$ are perspective from the point C and find the line from which they are perspective.

b) Find two other triangles, a point, and a line such that the two triangles are perspective from both the point and the line.