5.6 FRACTALS

Historically, geometry has focused on relatively simple, ideal shapes: circles, triangles, polyhedra, and the like. However, even a cursory glance at nature reveals a vast array of shapes unrelated to these traditional objects. Benoit Mandelbrot, the originator of fractals, found geometric structure underlying complicated natural shapes. In 1975 he coined the word fractal to describe the convoluted curves and surfaces that can be used to model natural shapes that had previously seemed beyond mathematical study.

Mathematicians initiated the abstract study of curves related to fractals before 1900. In 1904 Helge von Koch defined the Koch curve (Fig. 5.45) as the limit of an infinite process, illustrated in Fig. 5.46. Starting with the motif at the top of Fig. 5.46, we replace

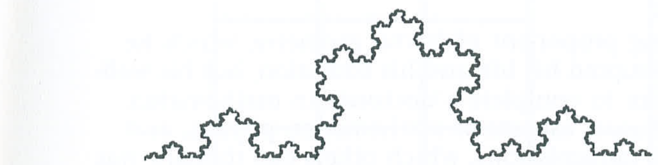

Figure 5.45 The Koch curve.

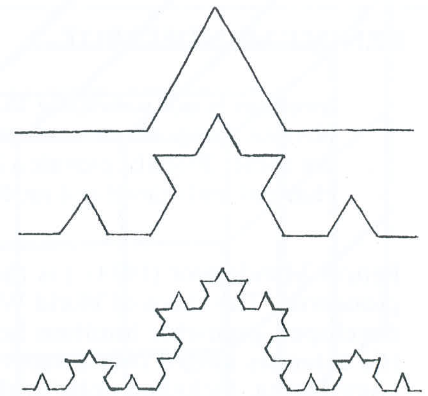

Figure 5.46 Iterations leading to the Koch curve.

each of its line segments with smaller copies of the motif. The middle curve shows the first *iteration*, and the bottom curve shows the second iteration. After infinitely many iterations we arrive at the Koch curve, which has the property of *self-similarity*: The entire curve is similar to a part of the curve.

Example 1 Show the Koch curve is infinitely long.

Solution. Suppose that the original motif in Fig. 5.46 has a length of 1 unit. After one iteration, there are four copies of the motif, each one-third as long, for a length of $4/3 \approx 1.3$ units. The second iteration has length $(4/3)^2 \approx 1.8$ units, and, in general, the nth iteration has length $(4/3)^n$ units, increasing in length as n increases. The length of the Koch curve is $\lim_{n\to\infty}(4/3)^n = \infty$. Note that the Koch curve is enclosed in a finite area despite its infinite length. ●

Exercise 1 Explore why the method of IFS in Section 4.4 produces the same curve as the method Koch used.

Mandelbrot noticed that many real phenomena, such as coastlines, mountains, and lungs, have a roughly self-similar shape: The smaller features of these objects have the same overall bumpiness as the larger features. Of course, no part of the coastline of France will exactly replicate the entire coastline. Furthermore, no real shape can exhibit even approximately self-similar shape at the subatomic level. Thus exact self-similarity is too limited to model nature. Mandelbrot uses the term *statistical self-similarity* to describe approximate similarity over a range of scales. He avoids an exact mathematical definition of this concept because such a definition would apply only to mathematical objects, defeating his purpose. Computers can readily draw statistically self-similar shapes by modifying the iterations at smaller scales with randomly generated fluctuations. The resulting graphics often look strikingly realistic and support the usefulness of statistical self-similarity However, Mandelbrot realized the need for more

BENOIT MANDELBROT

> Intuition is not something that is given. I've trained my intuition to accept as obvious things which were initially rejected as absurd and I find everyone can do the same. [Fractals] provide a handle to representing nature, and intuition can be changed and refined and modified to include them. —*Benoit Mandelbrot*

Benoit Mandelbrot (1924–) is the leading proponent of fractal geometry, which he pioneered. The chaos of World War II disrupted his life and his education, but his well-developed geometric intuition helped him to complete a doctorate in mathematics. His interests ranged over a variety of unusual aspects of mathematics, physics, and engineering, including noise in electrical transmissions, which others had thought was simply random. Mandelbrot found that the frequency of noise measured in second-long intervals reassembled the frequency at longer intervals. Slowly he found other phenomena with a uniformity under change of scale, or what he called statistically self-similar or fractal. He collected examples of fractal behavior much the way naturalists collect specimens.

Mandelbrot's interest in the application of fractals is coupled with an intense interest in mathematical ideas, although he is much less interested in mathematical proof. He draws on the results of others, coupled with his remarkable visual intuition and stunning computer graphics, to build new mathematical ideas and conjectures. He helped pioneer the use of computers to draw fractals and to approach mathematics as an experimental field.

structure if the mathematics of self-similarity is to lead to new insights, not just interesting graphics.

Developed in 1919, the Hausdorff dimension provides a measure of how convoluted mathematical shapes are, and Mandelbrot modified it to measure real objects. Felix Hausdorff (1868–1942) noted a relationship between dimensions and the growth in the

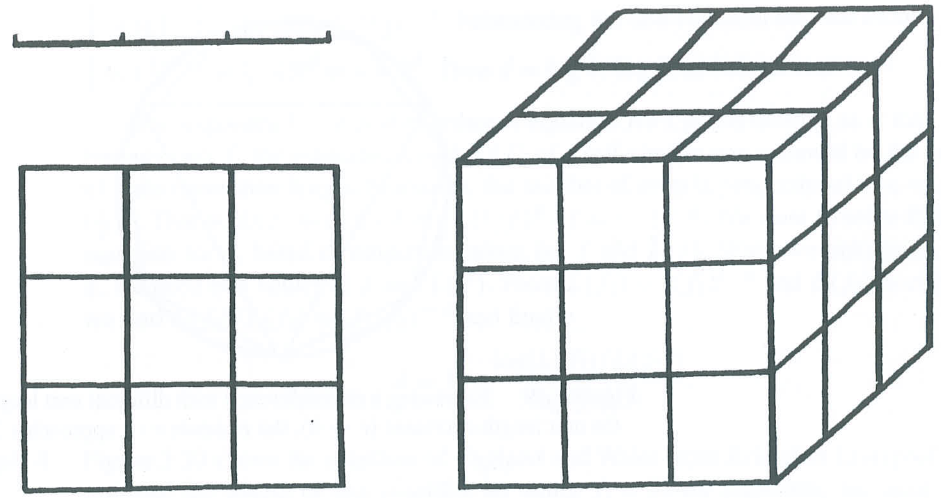

Figure 5.48

number of units as the measuring scale decreases. Figure 5.48 illustrates the intuition behind Hausdorff's approach. If we divide each side into three equal pieces, a line segment has $3 = 3^1$ smaller segments, a square has $9 = 3^2$ smaller squares, and a cube has $27 = 3^3$ smaller cubes. The dimension appears as the exponent. The smaller pieces are similar to the originals by a scaling ratio of $r = \frac{1}{3}$. (Hausdorff's technical definition uses analysis. See Falconer [5].)

Exercise 2 Create drawings analogous to Fig. 5.48, with each side divided into four smaller units, so $r = \frac{1}{4}$. Verify that the line segment has 4^1 smaller segments, the square has 4^2 small squares, and the cube has 4^3 smaller cubes.

We can generalize Fig. 5.48 and Exercise 2: The number of units is proportional to $n = (1/r)^d$, where d is the dimension and r is the scaling ratio. If we take logarithms and solve for d, we obtain the equation for the dimension:

$$d = \log n / \log(1/r). \quad (5.3)$$

Exercise 3 Verify that Eq. 5.3 for the dimension d holds for the line segment, square, and cube of Fig. 5.48 for any choice of r.

Hausdorff applied his notion to more interesting shapes than line segments, squares, and cubes. Thus, in Fig. 5.49, we can approximate the circumference of a circle with increasingly smaller units. As a curve, a circle is essentially a one-dimensional object. Thus halving the length of the units approximately doubles the number of units. None of the values $\log n / \log(1/r)$ exactly equals 1, but as r goes to 0, the limit equals 1. For simple shapes, such as a circle or even the Koch curve, Eq. 5.3 (or its limit as $r \to 0$) is sufficient to determine the Hausdorff dimension. More complicated sets of

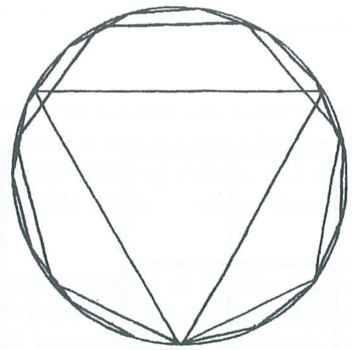

Figure 5.49 Estimating a circumference with different unit lengths. As the unit length decreases ($r \to 0$), the estimate $r \cdot n$ approaches $2\pi R$.

points require Hausdorff's exact definition. Using analysis, Hausdorff proved that every nonempty subset of $\mathbf{R}^k$ has a unique Hausdorff dimension of at most k.

Example 2 Find the Hausdorff dimension of the Koch curve.

Solution. In Example 1 a scaling factor of $r = \frac{1}{3}$ gives $n = 4$ times as many units. Then Eq. 5.3 gives $d = \log 4/ \log 3 \approx 1.262$. Verify that a scaling ratio $r = \frac{1}{9} = (\frac{1}{3})^2$ gives the same value of d. The Koch curve is too convoluted to be measured by the one-dimensional unit of length because it is infinitely long. However, the curve has an area of 0, so the curve isn't two-dimensional. The value $\log 4/ \log 3$ is the Hausdorff dimension, indicating that the convoluted Koch curve is between a line and a surface. ●

Unfortunately, the Hausdorff dimension doesn't apply to real shapes. In 1961, Lewis Richardson published a study of the estimated length of coastlines according to maps of different unit lengths. The estimated lengths differed widely, but Richardson found a function of the lengths in terms of the unit lengths. Mandelbrot recognized that Richardson's equation fits the intuition behind the Hausdorff dimension. (Richardson apparently was unaware of Hausdorff's work.) In Richardson's equation, we replace the original uninterpreted exponent with $1 - d$ to relate it to d, the Hausdorff dimension, to get

$$L(f) = c(f^{1-d}),$$

where f represents the unit length, $L(f)$ is the estimated length (or area) of the object for that value of f, and c is a constant, depending on the amount of the object measured.

Example 3 In the original motif of the Koch curve (see Fig. 5.46) let each segment have a length of $f_1 = \frac{1}{4}$. Then the four segments give $L(f_1) = 1$ as an estimated length of the Koch curve. In the first iteration when $f_2 = \frac{1}{3} \cdot \frac{1}{4}$, the 16 segments give $L(f_2) = \frac{4}{3}$. In the second iteration when $f_3 = \frac{1}{9} \cdot \frac{1}{4}$, $L(f_3) = \frac{16}{9}$, and so on. We solve for d in Richardson's equation by using two values of f. With f_1 we get $1 = c(\frac{1}{4})^{1-d}$, and with f_2 we get $\frac{4}{3} = c(\frac{1}{3} \cdot \frac{1}{4})^{1-d} = c(\frac{1}{3})^{1-d}(\frac{1}{4})^{1-d}$. Substituting the first equation into the second yields $\frac{4}{3} = (\frac{1}{3})^{1-d} = \frac{1}{3} \cdot (3)^d$ or $4 = 3^d$. Then $d = \log 4/ \log 3$, as before. ●

The exponent $1 - d$ in Richardson's equation isn't as mysterious as it may seem. For any unit f, the estimated length $L(f)$ of a self-similar curve should be the number of units times their length. Moreover, the number of units is proportional to $n = (\frac{1}{r})^d = (\frac{1}{f})^d$. That is, $L(f) = c \cdot n \cdot f = c \cdot (1/f)^d \cdot f = c \cdot f^{1-d}$. We want to solve Richard's equation for d, based on empirical values for f and $L(f)$. With two unknowns, c and d, we need two values of f and $L(f)$. From $L(f_1) = c(f_1)^{1-d}$ and $L(f_2) = c(f_2)^{1-d}$, we find $L(f_1)/L(f_2) = (f_1/f_2)^{1-d}$ and finally

$$d = 1 - \frac{\log[L(f_1)/L(f_2)]}{\log(f_1/f_2)}. \quad (5.4)$$

Example 4 Figure 5.50 shows the coastline of England and Wales from Bristol to Liverpool. If we estimate the length of this coastline by using $f_1 = 57$-mi segments, we need 5 segments, so $L(f_1) \approx 285$ mi. For $f_2 = 28.5$ mi, we need $12\frac{1}{2}$ segments, so $L(f_2) \approx 356$ mi; $f_3 = 14.25$ mi gives 32 segments and $L(f_3) \approx 456$ mi. As we shorten the unit length, we can follow the contour better, taking into account more and more of the multitude of peninsulas and bays. When we use f_1 and f_2, Eq. 5.4 gives $d = 1 - \log(285/356)/ \log(57/28.5) \approx 1.32$. ●

Exercise 4 Verify that $d \approx 1.36$ when you use f_2 and f_3 and $d \approx 1.34$ when you use f_1 and f_3.

The values of d that Richardson found for coastlines are remarkably stable over a range of scales f. (Such values can't be stable over all possible scales unless the object

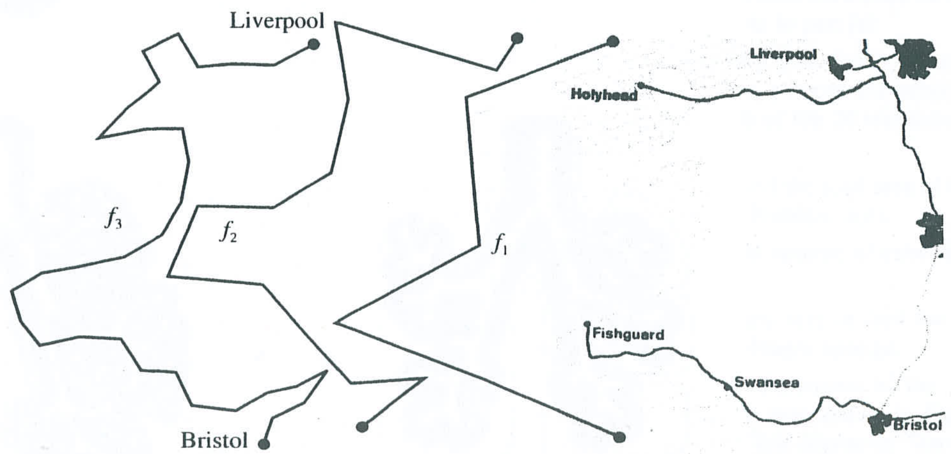

Figure 5.50 The mapped coastline of Wales and part of England and approximate outlines produced by using segments of different lengths.

is perfectly self-similar. The range of scales depends on the object studied.) Mandelbrot calls this empirical value the *fractal dimension* to distinguish it from Hausdorff's abstract definition. Mandelbrot and others have estimated fractal dimensions for a variety of natural curves and surfaces. He avoids defining a fractal, but the following provisional definitions are helpful. Example 2 illustrates how self-similar shapes, such as the Koch curve, can have unexpected dimensions. Self-similarity gives an exact number of copies, n, for an appropriate scaling ratio, r. More convoluted shapes generally have more small copies at a given scale and so have higher dimensions.

Provisional Definition A *fractal curve* has fractal dimension greater than 1. A *fractal surface* has fractal dimension greater than 2.

Fractals provide a good model for the lungs. The trachea splits into the bronchial tubes, which in turn split into shorter and narrower tubes. In addition, the embryonic development of the lung is an iterative process The convoluted surface of the lung greatly increases its area while keeping its overall volume small. The large surface is biologically essential because the amount of carbon dioxide and oxygen that the lungs

can exchange is roughly proportional to their surface area. Using a light microscope, biologists found approximately 80 m^2 of surface area in a lung (roughly the floor space of a small house). The higher magnification of an electron microscope yielded approximately 140 m^2. This increase in area at higher magnification corresponds to the measurements given in Example 4. Scientists have estimated the fractal dimension of a lung to be 2.17. Blood vessels, kidneys, the liver, and other organs have good fractal models.

Although fractals provide insightful models, scientists are hoping for more than explanations of already known facts. For example, why is the fractal dimension of a lung 2.17? After all, a higher dimension would give even more surface area. Perhaps higher dimensions impede the free passage of air or blood in the lung. Questions such as these provide ample challenges for research in the applications of fractals. (See Mandelbrot [12].)

PROBLEMS FOR SECTION 5.6

1. For each motif of the fractal curve given, sketch several iterations.
   - **a)** A stylized tree, where each branch splits into two others half as long.
   - **b)** A stylized tree, where each branch splits into three others half as long.
   - **c)** A Cantor set, whereby you divide a line segment into three equal pieces and remove the middle piece and iterate with the remaining pieces.
   - **d)** A modified Koch curve, with a square on the middle third of a line segment, rather than a triangle.
   - **e)** A modified Koch curve, whereby you divide a line segment into fourths and construct squares on the alternate sides of the two middle fourths.
   - **f)** A Sierpinski gasket, whereby you divide a triangle into four smaller triangles by connecting the midpoints of the sides, remove the middle triangle, and iterate with the remaining triangles.
2. Find the Hausdorff dimension of the fractals in Problem 1.
3. Investigate what happens to the figures and the fractal dimension when you increase the number of smaller copies in the fractals in Problem 1(b) and 1(e).
4. For each motif for a fractal surface given, sketch the first and second iterations and find the fractal dimension.
   - **a)** A "Koch pyramid," whereby you divide a triangle into four smaller triangles, as in 1(f), but construct a triangular pyramid on the middle triangle and iterate with each of the smaller triangles and the faces of the pyramid.
   - **b)** A "Koch cube," whereby you divide a unit square into nine smaller squares, construct a cube on the middle square, and iterate as in part (a).
   - **c)** A Menger sponge, whereby you divide a unit cube into 27 smaller cubes, remove the center cube, and iterate with each of the 26 remaining smaller cubes.
5.
   - **a)** Use a geometric series to find the total area of the infinitely many squares in Problem 1(d).
   - **b)** Repeat part (a) for the total volume of cubes in Problem 4(b).
   - **c)** Repeat part (a) for Problem 4(c) to find the volume remaining in the Menger sponge.
6. You can estimate the fractal dimension of the coastline of Norway using the map shown in Fig. 5.52 by using different length line segments. Start at Oslo and lay out first 1-in., then $\frac{1}{2}$-in., and then $\frac{1}{4}$-in. line segments along the coast to Bergen. Then use Eq. 5.4 to calculate the fractal dimension. On this map, 1 in. $\approx$ 50 mi.

Figure 5.52 The mapped coastline of southern Norway.

PROJECTS FOR CHAPTER 5

1. Design your own wallpaper patterns by hand or use software, such as TesselMania [9] under Suggested Media.
2. Which types of frieze patterns and wallpaper patterns can be formed with mirrors? Arrange the mirrors in various ways facing one another. Place symmetric and asymmetric designs in the region between the mirrors.
3. Investigate symmetry in the art of M. C. Escher. (See Schattschneider [16].)
4. Investigate symmetry in weaving. (See Pizzuto [14].)
5. Find examples of actual wallpaper and classify them. Certain symmetry types of wallpaper patterns predominate. Investigate the reasons, including aesthetic ones, that underlie the choice of symmetry in wallpaper.
6. Investigate symmetry in the art of various cultures. (Wade [20] provides a large variety of examples. For Islamic art, see El-Said and Parman [4].)
7. Investigate tilings of the plane, which generalize wallpaper patterns. (See Grünbaum and Shephard [8].)
8. Investigate the symmetries of continuous (nondiscrete) frieze and wallpaper patterns. Try to classify types of such patterns.
9. In many cultures circular friezes appear as ornaments on cups and other cylindrical objects. Circular friezes replace the translations of friezes with rotations about a vertical axis. The seven frieze groups become seven subgroups of $\overline{D'_n}$.
   - **a)** Draw a frieze with symmetry group **pmm2** on a sheet of paper. Roll the paper into a cylinder to create a circular frieze. Describe how the symmetries of the frieze become symmetries of the circular frieze. Explain why this circular frieze has $\overline{D'_n}$ for its group of symmetries. What determines the value of n that a circular frieze has? Why do the symmetries of other circular friezes form subgroups of some $\overline{D'_n}$?
   - **b)** For each of the six other types of friezes, make a corresponding circular frieze pattern. Describe the rotations, mirror reflections, and rotatory reflections of these circular friezes.
   - **c)** Which of the circular friezes in (b) has the symmetry group C_n? Which has the symmetry group D_n? (Actually, this group is called D'_nC_n for technical reasons.) Which has the symmetry group D'_n?
   - **d)** One of the remaining circular frieze has a horizontal mirror reflection. Its group is $\overline{C_n}$. Find it.
   - **e)** The remaining circular friezes have rotatory reflections of angles half as large as the rotations about the vertical axes. Find the one whose rotations are all around a vertical axis; this circular frieze has group $C_{2n}C_n$. Describe the additional rotations of the other, whose group is $D'_{2n}D'_n$.
   - **f)** Make a table to match the frieze and circular frieze groups.
   - **g)** Classify the group of an antiprism. (See Problem 2 of Section 5.4.)
   - **h)** Place a mirror face up on a table. Place two mirrors perpendicular to the first mirror, facing each other at various angles. Which of the circular frieze patterns can you make by placing symmetric and asymmetric designs in the region between the mirrors?
10. Investigate the symmetry of knots and braids.
11. Investigate symmetry in fugues and 12-tone music. (See Senechal and Fleck [18].)
12. Investigate color symmetry. (See Loeb [11].)
13. Borrow a set of hand bells to investigate symmetry and change ringing. (See Senechal and Fleck [18, 47].)
14. Investigate symmetry in crystals. (See Senechal [17].)
15. Build a Penrose tile by using the two types of tiles shown in Fig. 5.53. Match sides with dots on them to ensure that the tiling will be nonperiodic. Investigate Penrose tiles. (See Gardner [7], Grünbaum and Shephard [8], and Peterson [13].)

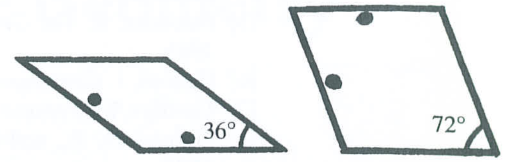

Figure 5.53 Two rhombi that can make a Penrose tiling. Match sides with dots and match sides without dots.

16. Investigate quasicrystals. (See Jaric [10], Peterson [13], and Senechal [17].)
17. Design fractals and investigate the Mandelbrot set and Julia sets. (See "The Desktop Fractal Design System" [4] under Suggested Media.)
18. Estimate the fractal dimension of real shapes. Determine the scale at which the fractal nature of your examples breaks down. The fractal dimension of a surface can be estimated by adding 1 to the fractal dimension of a typical cross section. (See Mandelbrot [12].)
19. Investigate the role of symmetry in quantum mechanics. (Bunch [1] and Rosen [15] provide elementary expositions and bibliographies.)
20. Investigate other ideas in symmetry. (See Bunch [1], Hargittai [9], Rosen [15], and Senechal and Fleck [18].)
21. Write an essay considering the relationship of symmetry and culture. Is a classification of designs by their symmetries culturally objective?
22. Write an essay discussing the application of abstract mathematics to the world around you. For example, why do proofs about infinite, perfect mathematical crystals tell you anything about real, finite crystals?
23. Write an essay discussing the notion of mathematics as an experimental science. Consider the following questions. Are proofs essential to mathematics? In what ways is experimental evidence appropriate in mathematics?

Suggested Readings

[1] Bunch, B. *Reality's Mirrors: Exploring the Mathematics of Symmetry*. New York: John Wiley & Sons, 1989.

[2] Crowe, D. *Symmetry, Rigid Motions, and Patterns*. Arlington, Mass.: COMAP, 1986.

[3] Crowe, D., and D. Washburn. Groups and geometry in the ceramic art of San Ildefonso. *Algebras, Groups and Geometries*, 1985, 2(3):263–277.
[4] El-Said, I., and A. Parman. *Geometric Concepts in Islamic Art*. Palo Alto, Calif.: Dale Seymour, 1976.
[5] Falconer, K. *The Geometry of Fractal Sets*. New York: Cambridge University Press, 1986.
[6] Gallian, J. *Contemporary Abstract Algebra*. Lexington, Mass.: D. C. Heath, 1994.
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[11] Loeb, A. *Color and Symmetry*. New York: John Wiley & Sons, 1971.
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[13] Peterson, I. *The Mathematical Tourist: Snapshots of Modern Mathematics*. New York: W. H. Freeman, 1988.
[14] Pizzuto, J. *101 Weaves in 101 Fabrics*. Pelham, N.Y.: Textile Press, 1961.
[15] Rosen, J. *Symmetry Discovered: Concepts and Applications in Nature and Science*. New York: Cambridge University Press, 1975.
[16] Schattschneider, D. *Visions of Symmetry: Notebooks, Periodic Drawings and Related Work of M. C. Escher*. New York: W. H. Freeman, 1990.
[17] Senechal, M. *Crystalline Symmetries: An Informal Mathematical Introduction*. Philadelphia: Bristol, 1990.
[18] Senechal, M., and G. Fleck (eds.). *Patterns of Symmetry*. Amherst: University of Massachusetts Press, 1977.
[19] Taylor, E., and J. Wheeler. *Spacetime Physics*, San Francisco: W. H. Freeman, 1966.
[20] Wade, D. *Geometric Patterns and Borders*. New York: Van Nostrand Reinhold, 1982.
[21] Wenninger, M. *Polyhedron Models*. New York: Cambridge University Press, 1971.
[22] Weyl, H. *Symmetry*. Princeton, N.J.: Princeton University Press, 1952.
[23] Yaglom, I. *Felix Klein and Sophus Lie: Evolution of the Idea of Symmetry in the Nineteenth Century*. Boston: Birkhäuser, 1988.

Suggested Media

1. "Adventures in Perception," 22-minute film, BFA Educational Media, Santa Monica, Calif., 1973.
2. "Chaos, Fractals and Dynamics: Computer Experiments in Mathematics," 63-minute video, American Mathematical Society, 1989.
3. "The Counting Theorem," 24-minute video, Films for the Humanities and Sciences, Princeton, N.J., 1996.
4. "The Desktop Fractal Design System," software and handbook by M. Barnsley, Academic Press, Boston, 1989.
5. "Dihedral Kaleidoscopes," 13-minute film, International Film Bureau, Chicago, 1966.
6. "Maurits Escher: Painter of Fantasies," $26\frac{1}{2}$-minute film, Coronet Films, Chicago, 1970.
7. "Symmetry Counts," 24-minute video, Films for the Humanities and Sciences, Princeton, N.J., 1996.
8. "Symmetries of the Cube," $13\frac{1}{2}$-minute film, International Film Bureau, Chicago, 1971.
9. "TesselMania," software, Minnesota Educational Computing Corporation, Minneapolis, 1995.