## 5.4 Symmetries in Higher Dimensions

### 5.4.1 Finite three-dimensional symmetry groups

Mathematicians and others since the time of ancient Greece have enjoyed the elegance of polyhedra, especially the five regular polyhedra. However, not until the nineteenth century was polyhedral symmetry analyzed. Auguste Bravais first published the classification of the finite three-dimensional symmetry groups (Theorem 5.4.2) in 1848, although Johann Hessel had proved it in 1830.

Some of the three-dimensional symmetry groups extend the families $\mathbf{C}_n$ and $\mathbf{D}_n$ discussed in Section 5.2. A prism (Fig. 5.29) has twice as many symmetries as the corresponding regular polygon that forms its top face. The group of symmetries of a prism is called $\overline{\mathbf{D}'_n}$, where $n$ is the number of vertices of the top face. Theorem 5.2.4 guarantees that half the symmetries in $\overline{\mathbf{D}'_n}$ are rotations. They form a group $\mathbf{D}'_n$, which is structurally the same as $\mathbf{D}_n$. In place of the $n$ mirror reflections of $\mathbf{D}_n$, $\mathbf{D}'_n$ has $n$ rotations of 180° around horizontal axes. The overbar in $\overline{\mathbf{D}'_n}$ and the other finite groups indicates a doubling of the size of the group to include mirror reflections and rotatory reflections. The groups $\mathbf{C}_n$ and $\mathbf{D}_n$ may be regarded as subgroups of $\overline{\mathbf{D}'_n}$, which also has other subgroups. (See Project 9.)

In addition to the subgroups of $\overline{\mathbf{D}'_n}$, there are a few other finite three-dimensional symmetry groups. These other groups all come from the symmetries of the regular polyhedra. Although there are five regular polyhedra, they determine only three different groups of symmetries. (See Fig. 1.44.) The tetrahedron has 24 symmetries, forming the tetrahedral group, $\overline{\mathbf{T}}$. These symmetries include rotations of 120° and 180°. The cube and octahedron have the same group of 48 symmetries, the octahedral group, $\overline{\mathbf{W}}$. The angles of rotation in this group are multiples of 90° and 120°. The icosahedron and dodecahedron have the same group of 120 symmetries, the icosahedral group, $\overline{\mathbf{P}}$. The angles of rotation in this group include multiples of 72°, 120° and 180°. These three groups have various subgroups, most notably their subgroups of rotations $\mathbf{T}$, $\mathbf{W}$, and $\mathbf{P}$. Many interesting polyhedra, including 11 of the 13 Archimedean solids have for their symmetry groups $\overline{\mathbf{T}}$, $\overline{\mathbf{W}}$, or $\overline{\mathbf{P}}$. The two remaining Archimedean solids have only rotations, and their groups of symmetries are $\mathbf{W}$ and $\mathbf{P}$. (See Wenninger [21] for pictures of these polyhedra.)

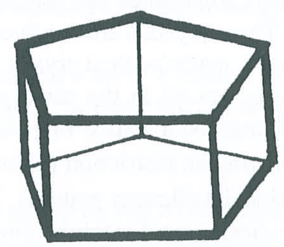

Figure 5.29

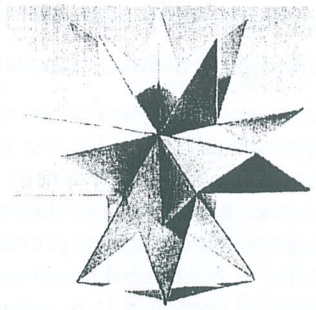

Figure 5.30

**Example 1** Find the group of symmetries of the great stellated dodecahedron (Fig. 5.30).

***Solution.*** This polyhedron has axes of rotation other than those of a prism. The axis facing out allows rotations of 72°, so the group can't be $\overline{\mathbf{T}}$ or $\overline{\mathbf{W}}$ or their subgroups. Because this polyhedron has mirror reflections, by Theorem 5.4.2 below, it must have $\overline{\mathbf{P}}$ for its group of symmetries. ●

**Theorem 5.4.1** The three-dimensional isometries in a finite symmetry group must fix a point and so form a subgroup of the symmetries of a sphere.

***Proof.*** See Problem 3. ■

**Theorem 5.4.2** A finite group of three-dimensional isometries must be one of the following or one of their subgroups: $\overline{\mathbf{D}'_n}$, $\overline{\mathbf{T}}$, $\overline{\mathbf{W}}$, and $\overline{\mathbf{P}}$.

***Proof.*** See Weyl [22, 149ff]. ■

### 5.4.2 The crystallographic groups

The beauty of crystals, especially gems, have fascinated people for centuries. However, prior to 1849 no chemical explanation existed of the visible regularities and other properties of crystals. The mathematical classification by Bravais in 1849 of the 32 types of crystals spurred the study of geometric arrangements of atoms in crystals. (Chemists sometimes say that there are 33 types; two are mirror reflections of each other.) These crystals are the three-dimensional analogs to frieze patterns and wallpaper patterns. A mathematical crystal is a discrete pattern having translations in at least three directions, not all in the same plane. The subgroup of translations takes a point to a three-dimensional lattice of points (Fig. 5.31). To classify crystals, Bravais used the crystallographic restriction (Theorem 5.3.3) which gives the rotations possible for two-dimensional wallpaper patterns. Theorem 5.3.3 applies to crystals as follows. Consider a three-dimensional rotation about an axis $k$ and a point $P$ in the lattice not on $k$. The rotation moves $P$ to another point in the plane perpendicular to $k$ and through $P$. The



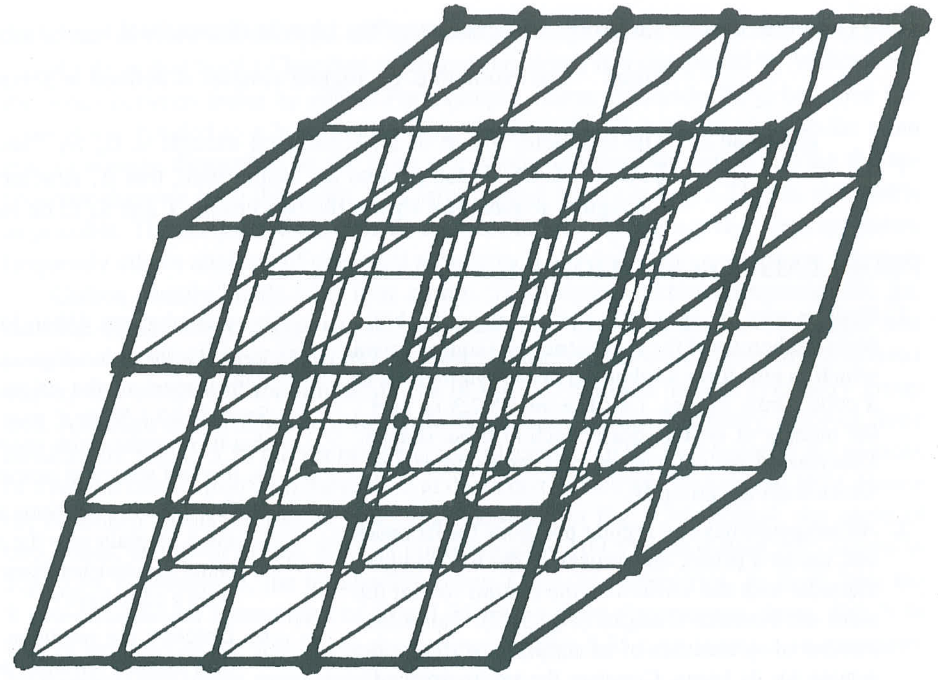

Figure 5.31

points of the lattice lying in this plane form a wallpaper pattern. Thus the rotation must be a rotation of the wallpaper pattern. Hence Theorem 5.3.3 applies to mathematical (and chemical) crystals. However, crystals can combine these possible rotations in more ways than wallpaper patterns can. (See Senechal [17].)

### 5.4.3 General finite symmetry groups

Finite groups appear in many contexts, many of which don't correspond in any obvious way to isometries in Euclidean geometry in any number of dimensions. Nevertheless, any finite group may be thought of as a symmetry group of some figure. In effect, we may think of the group as a transformation group moving points around. In particular Theorem 5.4.4 allows us to think of these points as embedded in Euclidean space of an appropriate dimension and the transformations of the group as isometries.

**Definition 5.4.1** The group of all transformations on a set of $n$ distinct points is the *symmetric group* $\mathbf{S}_n$. If $n$ is a positive integer, $n!$ (read "$n$ factorial") is the product of the numbers from 1 to $n$.

**Theorem 5.4.3** For any positive integer $n$, the group $\mathbf{S}_n$ has $n!$ elements.

***Proof.*** See Problem 7. ■

**Theorem 5.4.4** The group of isometries of the $(n-1)$-dimensional regular simplex is $\mathbf{S}_n$.

*Proof.* See Problem 8. (A regular simplex is defined in Section 2.5.) ■

**Example 2** The symmetry group of an equilateral triangle is $\mathbf{D}_n$ by Theorem 5.2.2 and is $\mathbf{S}_3$ by Theorem 5.4.4. These groups are isomorphic, that is, structurally identical. Similarly, the regular tetrahedron shows the two groups $\overline{\mathbf{T}}$ and $\mathbf{S}_4$ to be isomorphic. ●

## PROBLEMS FOR SECTION 5.4

1. Explain why the symmetries of a rectangular box form a subgroup of the symmetries of a square prism, which in turn form a subgroup of the symmetries of a cube. Draw figures. Use Theorem 5.2.3 to find the number of symmetries of each of these shapes. Describe all the symmetries of a rectangular box, which form the group $\overline{\mathbf{D}}'_2$.
2. An *antiprism* has two regular polygons for its bases, but, unlike a prism, the vertices of the top polygon alternate with the vertices on the bottom so that the sides are isosceles triangles (Fig. 5.32). Count the number of symmetries of an antiprism with regular $n$-gons for its bases. Compare the symmetries of an antiprism with the symmetries of the prism with regular $n$-gons for bases.

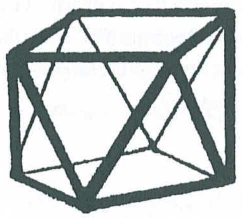

Figure 5.32

3. Prove Theorem 5.4.1. [*Hint:* See the proof of Theorem 5.2.1.]
4. Show that the tetrahedral group is a subgroup of the octahedral group. [*Hint:* Fit a tetrahedron in a cube.]
5. Let $\zeta$ be the three-dimensional central symmetry with respect to the origin. (See Problem 2 of Section 4.5.) Write $\zeta$ as a $3 \times 3$ matrix and show that it commutes with every spherical isometry. That is, if $\sigma$ is a spherical isometry, $\sigma \circ \zeta = \zeta \circ \sigma$.
6. a) Describe the rotations and mirror reflections of a cube. Explain why the regular octahedron has the same symmetries. How many rotatory reflections does a cube have?

   b) Describe the rotations and mirror reflections of a regular tetrahedron. How many rotatory reflections does a tetrahedron have?

   c) Repeat part (a) for a regular icosahedron and dodecahedron.
7. Use induction to prove Theorem 5.4.3. [*Hint:* For the set $\{1, 2, \ldots, n\}$, why are there $n$ possible places where 1 can go? Use the reasoning followed in Theorem 5.2.3.
8. Prove Theorem 5.4.4.
9. Classify the symmetry groups of the 13 semiregular polyhedra (the Archimedean solids). (See Wenninger [21] for pictures of these polyhedra.)
10. a) For which $n$ is the group $\overline{\mathbf{D}}'_n$ a subgroup of $\overline{\mathbf{T}}$?

    b) Repeat part (a) but replace $\overline{\mathbf{T}}$ with $\overline{\mathbf{W}}$.

    c) Repeat part (a) but replace $\overline{\mathbf{T}}$ with $\overline{\mathbf{P}}$.

    d) Repeat part (a) but replace $\overline{\mathbf{T}}$ with $\overline{\mathbf{D}}'_k$, where $n \leq k$.

## 5.5 SYMMETRY IN SCIENCE

### 5.5.1 Chemical structure

Chemists benefit greatly from a geometric understanding of the arrangement of the atoms (and ions) in chemical compounds. (For simplicity we refer to the parts of compounds as *atoms*, ignoring the distinction between atoms and ions. Similarly, we avoid