5.3 Symmetry in the Plane

Although real designs cannot contain infinitely many copies of a motif, many designs from various cultures convey that impression. To analyze these patterns we assume that the motif does repeat infinitely often, with translations either in just one direction

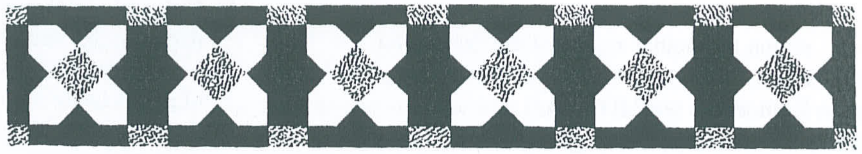


Figure 5.15 An Indian frieze pattern.

(Fig. 5.15) or in more than one direction (Fig. 5.16). The classification of the finitely many symmetry groups for frieze and wallpaper patterns provides anthropologists and archaeologists a way to analyze designs of other cultures that transcends cultural boundaries.

Definition 5.3.1 Repetitions of a bounded motif are *discrete* iff there is a minimum positive distance between translations of the motif. A *frieze pattern* is a discrete plane pattern that has translations in just one direction as symmetries. A *wallpaper pattern* is a discrete plane pattern that has translations in more than one direction as symmetries.

To classify frieze patterns we first find which isometries can be symmetries of frieze patterns and then determine how they combine to form groups of symmetries. WLOG we assume that the translations are always horizontal. By Problem 4 frieze patterns

Figure 5.16 A Bornean wallpaper pattern.

always have a horizontal line that must be stable under every symmetry of the frieze. This line, which we call the *midline*, provides one way to limit the possible isometries for frieze patterns. The frieze pattern shown in Fig. 5.15 has all the symmetries described in Theorem 5.3.1, showing that all of them are possible.

Theorem 5.3.1 The only symmetries of a frieze pattern with horizontal translations are horizontal translations, vertical mirror reflections, glide reflections, and mirror reflections over the midline of the frieze pattern, and rotations of 180° with centers on the midline of the frieze pattern.

Exercise 1 Draw figures to illustrate the proof of Theorem 5.3.1.

Proof. Show that all other isometries map the midline to a different line and so cannot be symmetries of the frieze pattern. Translations in a direction other than horizontal lift or lower the midline, so they are eliminated. Rotations other than 180° (or 0°, the identity) tilt the midline. Rotations of 180° whose centers are not on the midline shift it to a different horizontal line. Mirror and glide reflections over lines that are not vertical or horizontal also tilt the midline. Mirror and glide reflections over horizontal lines other than the midline, as well as glide reflections over vertical lines, shift the midline to a different horizontal line. By Theorem 4.2.7, we have eliminated all other isometries.

The analysis of how these possible symmetries fit to form groups requires a deeper understanding of groups than needed in Section 5.2 because now the groups are infinite. (We often simply say *group* instead of symmetry group or transformation group.) We need to find a small number of symmetries for each frieze pattern from which we can generate all the others.

Definition 5.3.2 A *subgroup* **H** of a group **G** is a subset of **G** that is a group in its own right, using the same operation as **G**. The elements $g_1, g_2, \ldots, g_n$ of **G** *generate* a subgroup **H** iff these elements are in **H** and every element h of **H** can be written in terms of these elements and their inverses, in some order and with any finite number of repetitions. We write $\langle g_1, g_2, \ldots, g_n \rangle$ to indicate the subgroup generated by these elements.

Example 1 Show that the dihedral group $\mathbf{D}_n$ is generated by two neighboring mirror reflections, μ_1 and μ_2.

Solution. The composition of μ_1 and μ_2 gives the smallest rotation: $\mu_1 \circ \mu_2 = \rho$. Repetitions of this composition, for example, $\mu_1 \circ \mu_2 \circ \mu_1 \circ \mu_2 = \rho^2$, give other rotations. (See Fig. 5.8.) We generate the mirror reflections as compositions of the form $\mu_1 \circ \rho^i = \mu_1 \circ (\mu_1 \circ \mu_2)^i$, for some power i. ●

Exercise 2 Explain why one translation generates the group of symmetries T of the frieze pattern shown in Fig. 5.17.

Example 2 A translation, a vertical mirror reflection, and the horizontal mirror reflection generate the group of symmetries of the frieze pattern shown in Fig. 5.15.

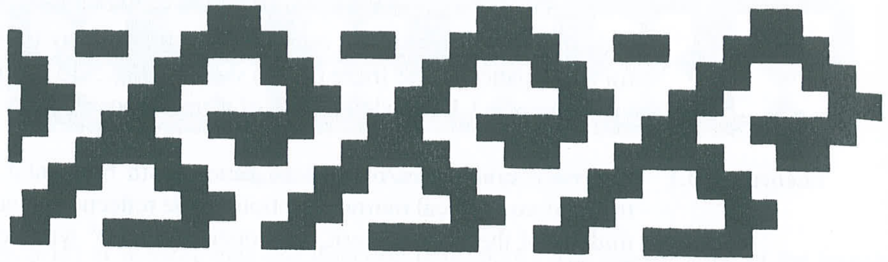


Figure 5.17 A Mexican frieze pattern.

Solution. We use properties from Chapter 4 to analyze compositions. The smallest translation generates all the others. The compositions of a vertical mirror reflection with the translations give all the vertical mirror reflections. Similarly, the horizontal mirror reflection and the translations generate the glide reflections. The composition of the vertical and horizontal mirror reflections is a rotation of 180°. The other rotations are obtained by composing that rotation with the translations. ●

Theorem 5.3.1 reveals that the symmetry group for the frieze shown in Fig. 5.15 is in some sense the largest such group. The group in Exercise 2 must in the same sense be the smallest, because every frieze pattern must have translations and this frieze pattern has only translations. To be completely rigorous, we need to be more careful. Let T' be the translations that shift each motif an even number of positions in Fig. 5.17. Then T' is a group of symmetries even smaller than group T in Exercise 2. However, T' doesn't differ in any substantial way from T. In algebraic terms, the groups are isomorphic. (See Section 1.4.) The geometric difference is that the distance between repetitions using T' is twice the distance between repetitions using T, which is irrelevant for finding different types of frieze patterns. WLOG we assume that any two friezes have the same smallest translation to the right. Theorem 5.3.2 shows that there are just seven types of frieze patterns.

Example 3 Figure 5.18 shows the seven types of frieze patterns, each having a group of symmetries different from the others. Beside each pattern is the name of the group of symmetries for that pattern. The names of the groups **pxyz** tell us what symmetries they have. If **x = m**, there are vertical mirror reflections. If **y = m**, there is a horizontal mirror reflection, and, if **y = g** there are horizontal glide reflections but not a horizontal mirror reflection. If **z = 2**, there are rotations of 180°. A **1** in any of these positions indicates that the group doesn't have this type of symmetry. ●

Theorem 5.3.2 There are exactly seven groups of symmetries for frieze patterns, up to isomorphism.

Proof. Example 3 shows that there are at least seven frieze groups. To show there are no others, we consider the possible sets of generators for frieze groups chosen from the isometries described in Theorem 5.3.1. We use τ for the smallest translation to the right,

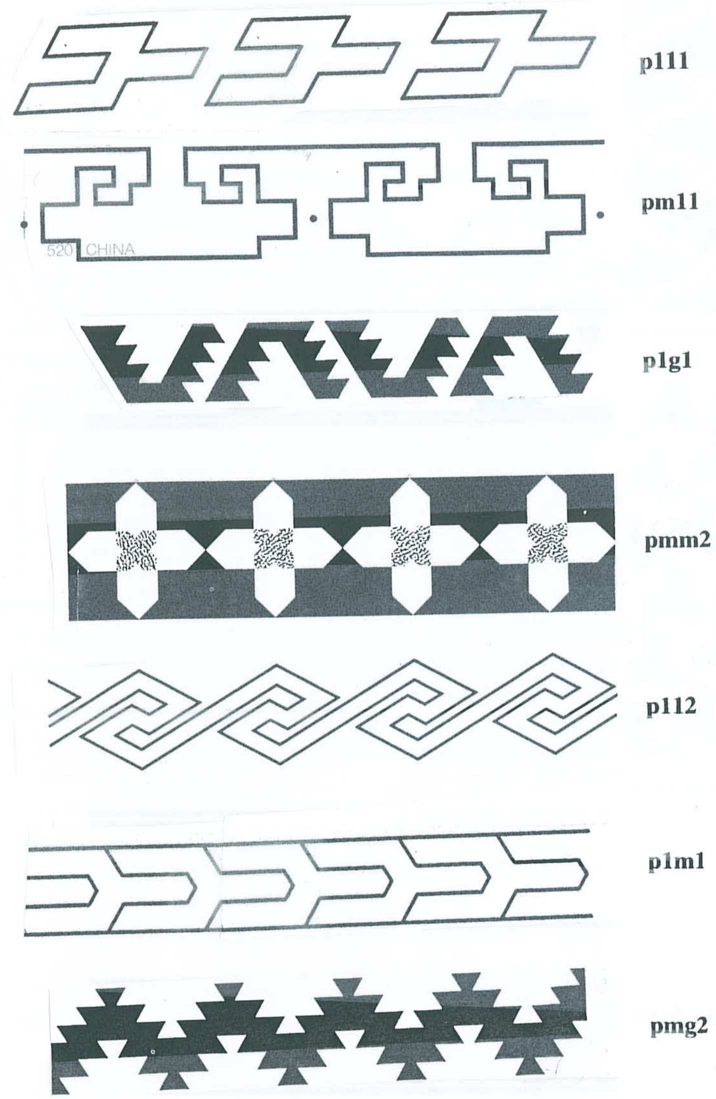


Figure 5.18

ρ for a rotation, η for the horizontal mirror reflection, ν for a vertical mirror reflection, and γ for a glide reflection.

By Problem 6, we do not need to consider every possible such set because of the following observations. All possible rotations are generated by any one rotation and τ. All possible vertical mirror reflections are generated by any one vertical mirror reflection and τ. The horizontal mirror reflection (or a glide reflection) and τ generate all the glide reflections. Finally, the composition of a vertical mirror reflection and a rotation can give two different types of symmetry. If the center of rotation is on the line of reflection, the composition is the horizontal mirror reflection. Otherwise the composition is a glide reflection.

Thus we need to consider generators of the form $\langle \tau, ? \rangle$, where ? is replaced by some of the following general symmetries: ρ, ρ', ν, η, or γ, where we assume the center of ρ to be on the line of reflection of ν but the center of ρ' not to be. (If ν is not one of the generators, it does not matter whether we use ρ or ρ'.) Then $\langle \tau \rangle = \mathbf{p111}$, $\langle \tau, \rho \rangle = \mathbf{p112}$, $\langle \tau, \nu \rangle = \mathbf{pm11}$, $\langle \tau, \eta \rangle = \mathbf{p1m1}$, and $\langle \tau, \gamma \rangle = \mathbf{p1g1}$. We obtain **pmg2** from $\langle \tau, \rho', \nu \rangle$, $\langle \tau, \rho, \gamma \rangle$, $\langle \tau, \nu, \gamma \rangle$, or $\langle \tau, \rho', \nu, \gamma \rangle$. We obtain **p1m1** from $\langle \tau, \eta, \gamma \rangle$ or $\langle \tau, \eta \rangle$. In Problem 6 you are asked to show that all other sets of generators yield **pmm2**. ∎

The classification of wallpaper patterns is more complicated than that of the frieze patterns. Figure 5.19 shows that different rotations are possible. However, Theorem 5.3.3 reveals that the angles of rotation illustrated in Fig. 5.19 are the only possible angles. This result is called the crystallographic restriction because it is also crucial in the classification of three-dimensional crystals.

Theorem 5.3.3 **The Crystallographic Restriction** The minimal positive angles of rotations that can be symmetries of a wallpaper pattern are 60°, 90°, 120°, and 180°, and 360°. All other angles of rotations for a given wallpaper pattern are multiples of the minimum angle.

Proof. Let A be a center of rotation for a wallpaper pattern and let B be a point closest to A for which some symmetry takes A to B. Then B must, by symmetry, also be a center of rotation for the wallpaper pattern with the same angles as at A. No two other images of A can be any closer together than are A and B. WLOG assume that A is to the left of B. Let ρ be the smallest positive rotation with center at A and ϕ be the smallest negative rotation with center at B. Now consider $A' = \phi(A)$ and $B' = \rho(B)$ (Fig. 5.20). By symmetry, both A' and B' must be centers of rotation like A. If ρ rotates less than 60°, then $d(A', B')$ will be less than $d(A, B)$, which is impossible. Similar reasoning (see Problem 7) eliminates other minimum angles except 90°, 120°, 180°, or 360°. Hence only the specified angles are compatible with wallpaper patterns. ∎

Theorem 5.3.4 There are exactly seventeen groups of symmetries for wallpaper patterns, up to isomorphism.

Proof. See Crowe [2]. ∎

The Russian chemist and mathematician Vyatseglav Fedorov in 1891 first stated and proved Theorem 5.3.4, but his proof wasn't widely noted. Several other mathematicians, including Felix Klein, independently found and proved this classification. The proof of Theorem 5.3.4 is based on group theory, but it isn't difficult to suspect geometrically that mirror and glide reflections can fit particular angles of rotation in only finitely many ways. Thus the number of wallpaper patterns is finite, even if the number 17 remains somewhat mysterious. The flowchart presented in Fig. 5.21 compresses the mathematics of the proof into a methodical way of classifying wallpaper patterns. The geometer Don Crowe developed such flow charts to aid archaeologists and anthropologists.

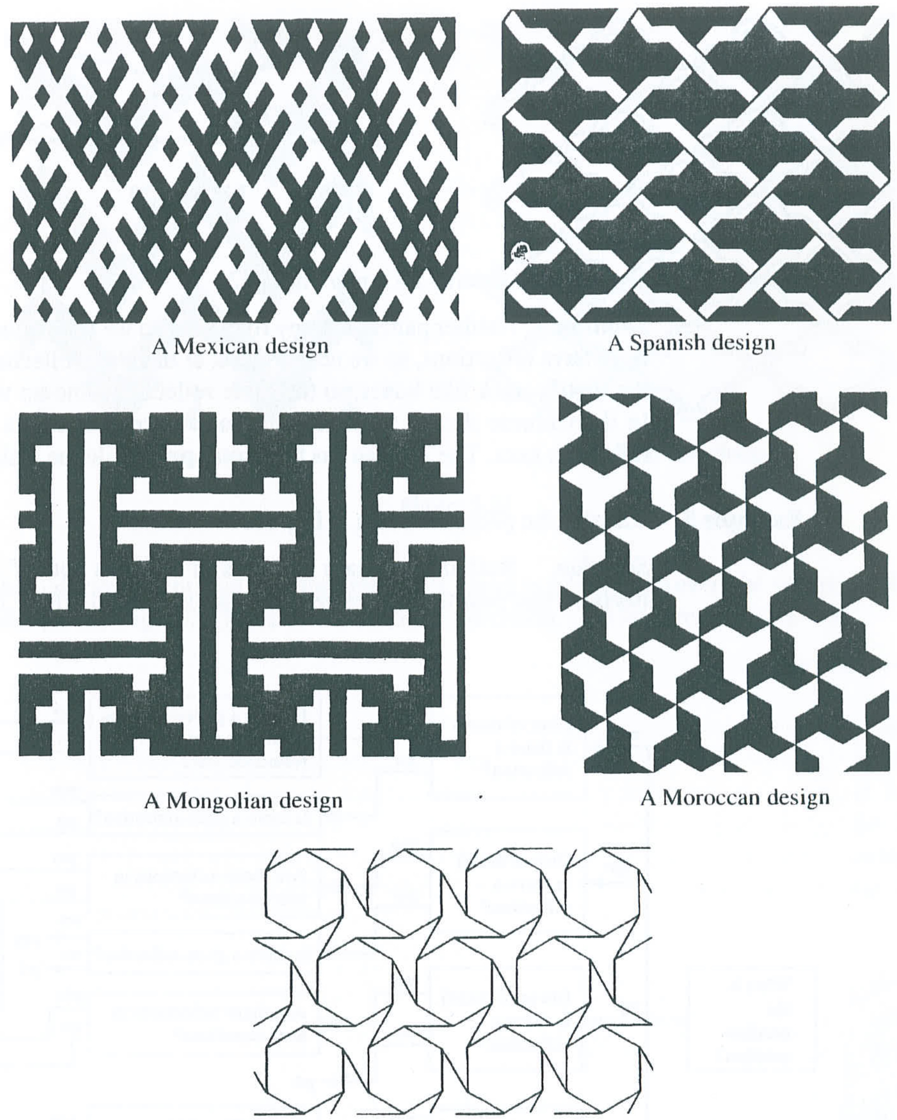


Figure 5.19 Wallpaper patterns.

The names of the wallpaper groups in Fig. 5.21 aren't as simple as the names of the frieze groups. The numbers **2**, **3**, **4**, and **6** refer to the maximum number of rotations around a center of rotation, as do the groups $\mathbf{C}_n$ and $\mathbf{D}_n$. The letters **m** and **g** refer to mirror and glide reflections. The letter **c** stands for a rhombic lattice, instead of a rectangular lattice. The difference is explained in Example 4. Then Example 5 considers two groups that are often as difficult to distinguish as are their names, **p31m** and **p3m1**.

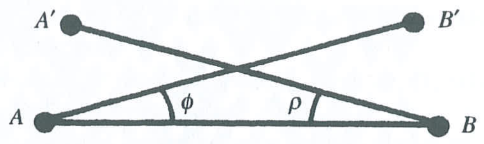


Figure 5.20

Example 4 Classify the patterns shown in Fig. 5.22.

Solution. Neither pattern has any rotations, so we follow the *none* branch in Fig. 5.21. Both have reflections, so we need to look at the glide reflections. In the Zairean design, the motifs stack like boxes, so the glide reflections line up with the mirror reflections. In the Chinese design the motifs alternate, like bricks in a wall, enabling new glide reflection axes. The Zairean has the group **pm**, while the Chinese has **cm**. ●

Example 5 Classify the patterns shown in Fig. 5.23.

Solution. Both designs have rotations of 120° but not 60°. Each has some mirror reflections that pass through centers of rotation. Indeed, Fig. 5.23(a) has mirror reflections

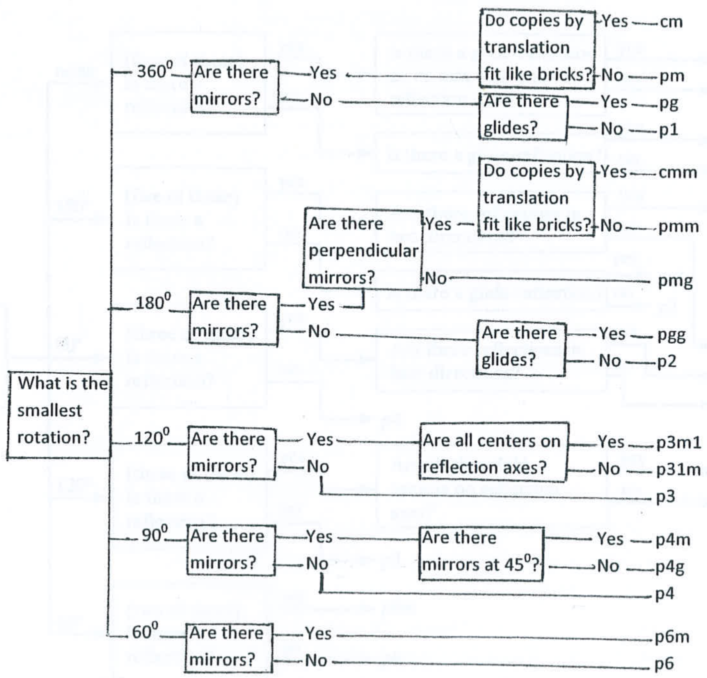
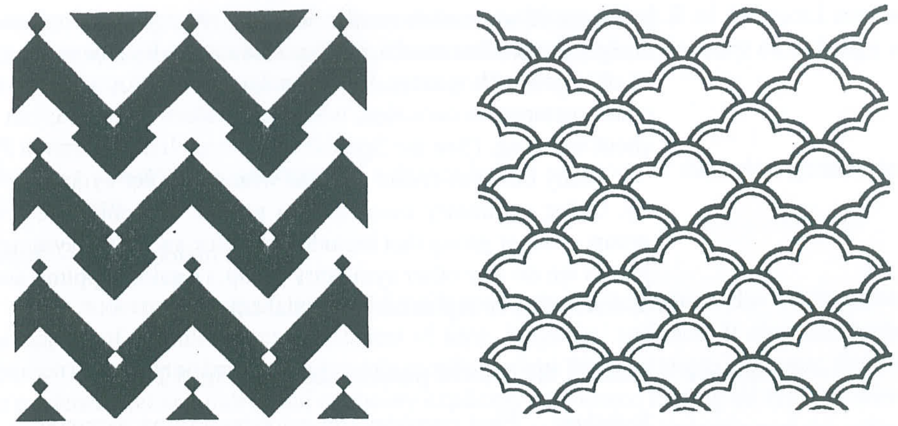


Figure 5.21 Flow chart for the 17 wallpaper groups.



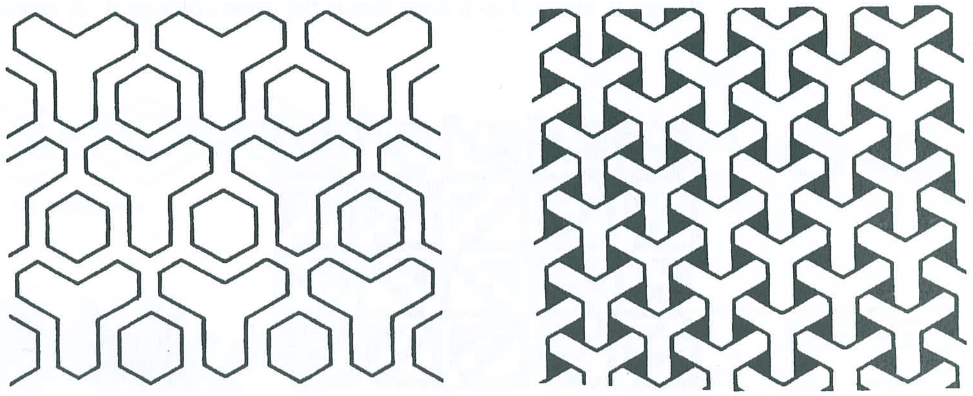
A Zairean design

A Chinese design

Figure 5.22

through every center of rotation. Hence its group is **p3m1**. However, the centers of the triangles in Fig. 5.23(b) don't have mirror reflections, so it has group **p31m**. ●

The anthropologist Dorothy Washburn teamed with the geometer Don Crowe to pioneer the use of symmetry groups in cross-cultural studies. Before their work, researchers had tried to analyze the varying motifs of different cultures. Although some characteristics seem apparent, the great variety of motifs made an analysis of motifs culturally subjective. However, the symmetry groups are independent of culture. For many cultures, the types of frieze and wallpaper designs used by their artists remain the same over long periods of time. This stability provides people studying cultures another marker in the study of societies and their interactions. A new design can indicate the



(a)

(b)

Figure 5.23 Two Japanese designs.

influence of trade with another region. Cultures that emphasize weaving tend to utilize designs from other media, such as ceramics, which have the symmetry patterns that can be obtained with weaves. For an ancient culture for which no traces of weaving may remain, patterns on ceramics, which can endure millennia, can provide indirect evidence about weaving. (See the Spanish and Mongolian patterns in Fig. 5.19.)

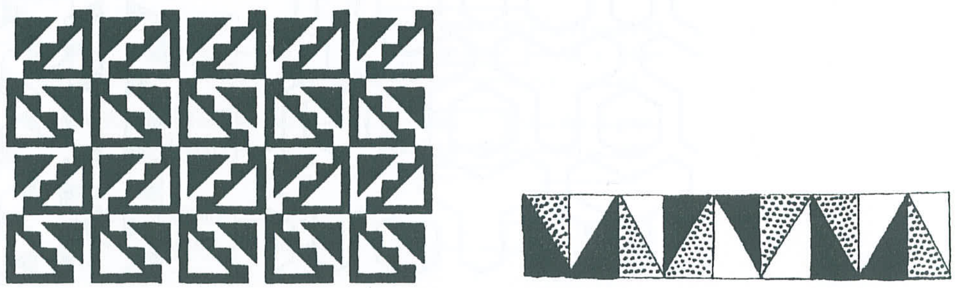
Many cultures create patterns with two-color symmetry or multiple-color symmetry. Color symmetry involves two groups: the color-preserving group and the color group, a larger group that includes the color-switching symmetries. We can analyze each just as we do any other symmetry group. (Again, stippling and cross-hatching are used, as necessary, to represent additional colors.)

Example 6 Classify the colored patterns shown in Fig. 5.24.

Solution. First consider the color-preserving symmetries shown in Fig. 5.24(a), a Peruvian design. All the black "staircases" are upright, implying no color-preserving rotations. The rows of black staircases alternate facing left and right, indicating no mirror reflections but indicating glide reflections preserve colors. From Fig. 5.21, the color-preserving group is **pg**. All the white staircases are upside down, so 180° rotations can switch colors. Horizontal mirror reflections between rows switch colors, but no vertical mirrors work. Figure 5.21 then tells us the color group is **pmg**. We write the pair of groups with the color group on top: **pmg/pg**.

The pattern shown in Fig. 5.24(b) is a three-color frieze pattern. The central vertical mirror reflection preserves the black parts but switches the white and stippled parts. The only symmetries preserving all three colors are translations and glide reflections, so the color-preserving group is **p1g1**. The color-switching group includes vertical mirror reflections and rotations, but not horizontal mirror reflections. Hence the color group is **pmg2**, and the classification is **pmg2/p1g1**. Note that some translations in **pmg2** are shorter than the translations that preserve colors. ●

Definition 5.3.3 A symmetry of a design is *color-preserving* iff every repetition A of the motif in the design is mapped to a copy that is the same color as A. A symmetry κ of a design is



(a) (b)

Figure 5.24

a *color-switching symmetry* iff, whenever the repetitions A and B of the motif are the same color, then $\kappa(A)$ and $\kappa(B)$ are the same color. A *color symmetry* of a design is either a color-preserving or a color-switching symmetry.

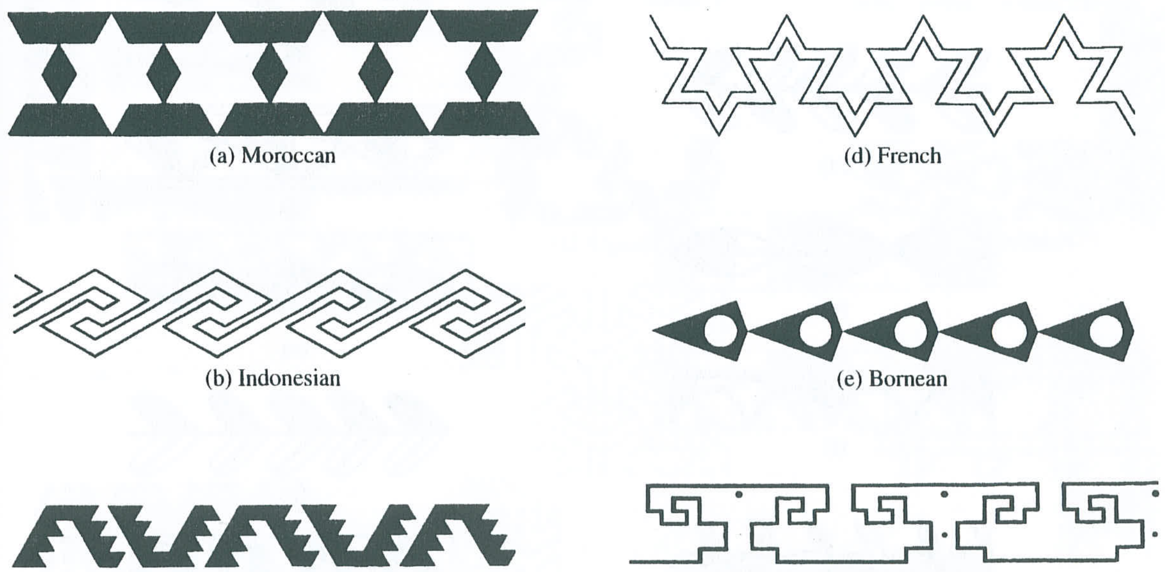
Theorem 5.3.5 The color-preserving symmetries of a design form a subgroup of the color symmetries of the pattern.

Proof. See Problem 8. ■

The widespread use of frieze and wallpaper patterns throughout the world shows the appeal of symmetry across barriers of time, language, and race. It also shows the geometric understanding needed to join motifs to make these patterns. However, there is no evidence that any individual or society explicitly considered finding all types of these patterns until the late nineteenth century. It is no accident that Fedorov and the others who thought about classification were trained in formal mathematics, especially group theory. Mathematics provides new ways to see the world, enriching understanding.

PROBLEMS FOR SECTION 5.3

1. Classify the frieze patterns shown in Fig. 5.25.



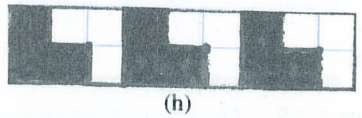
(a) Moroccan (d) French

(b) Indonesian (e) Bornean

(c) Peruvian (f) Chinese

Figure 5.25

2. Classify the two-color frieze patterns shown in Fig. 5.26. These designs represent all 17 types of two-color frieze patterns.



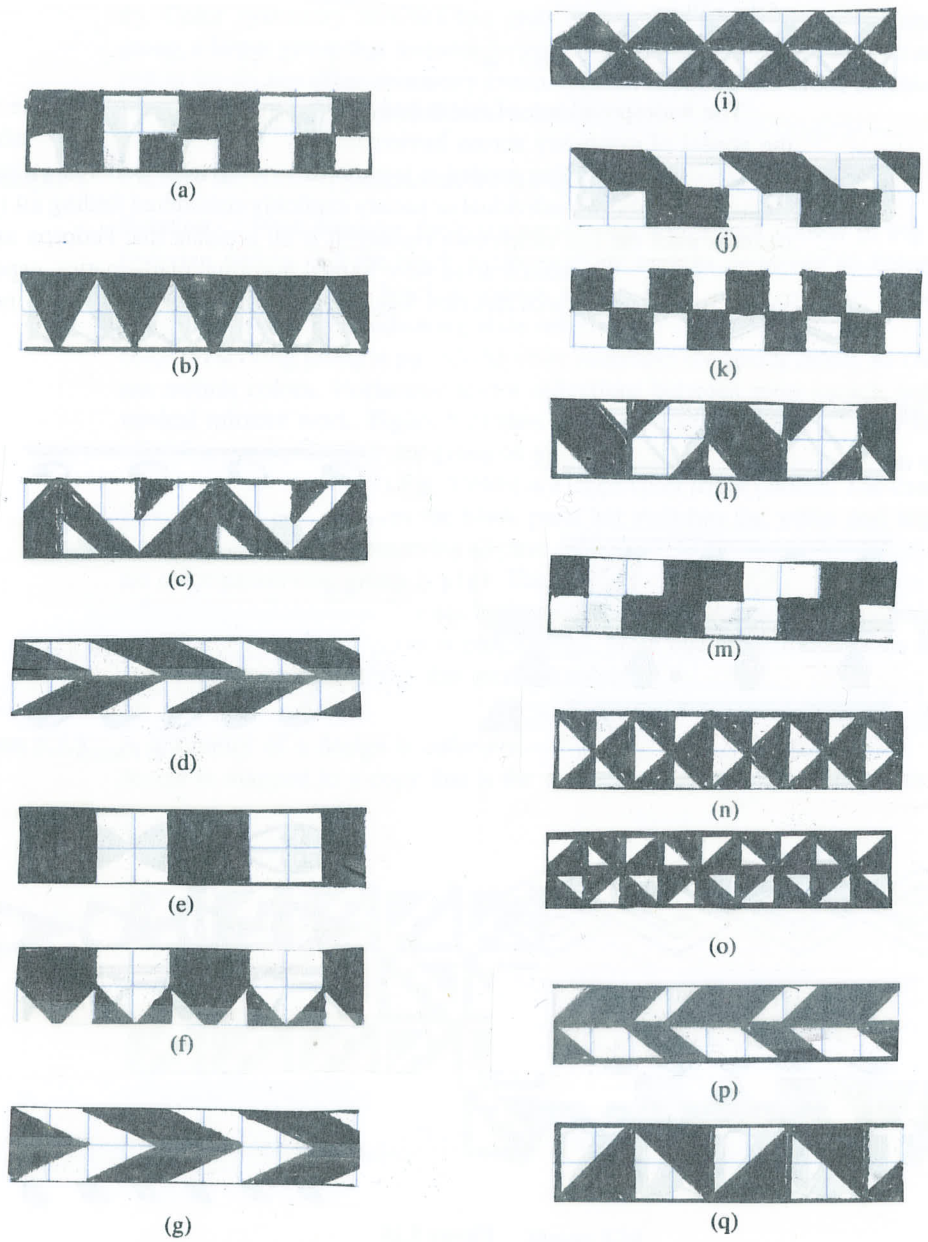


Figure 5.26

3. Classify the wallpaper patterns shown in Fig. 5.27.

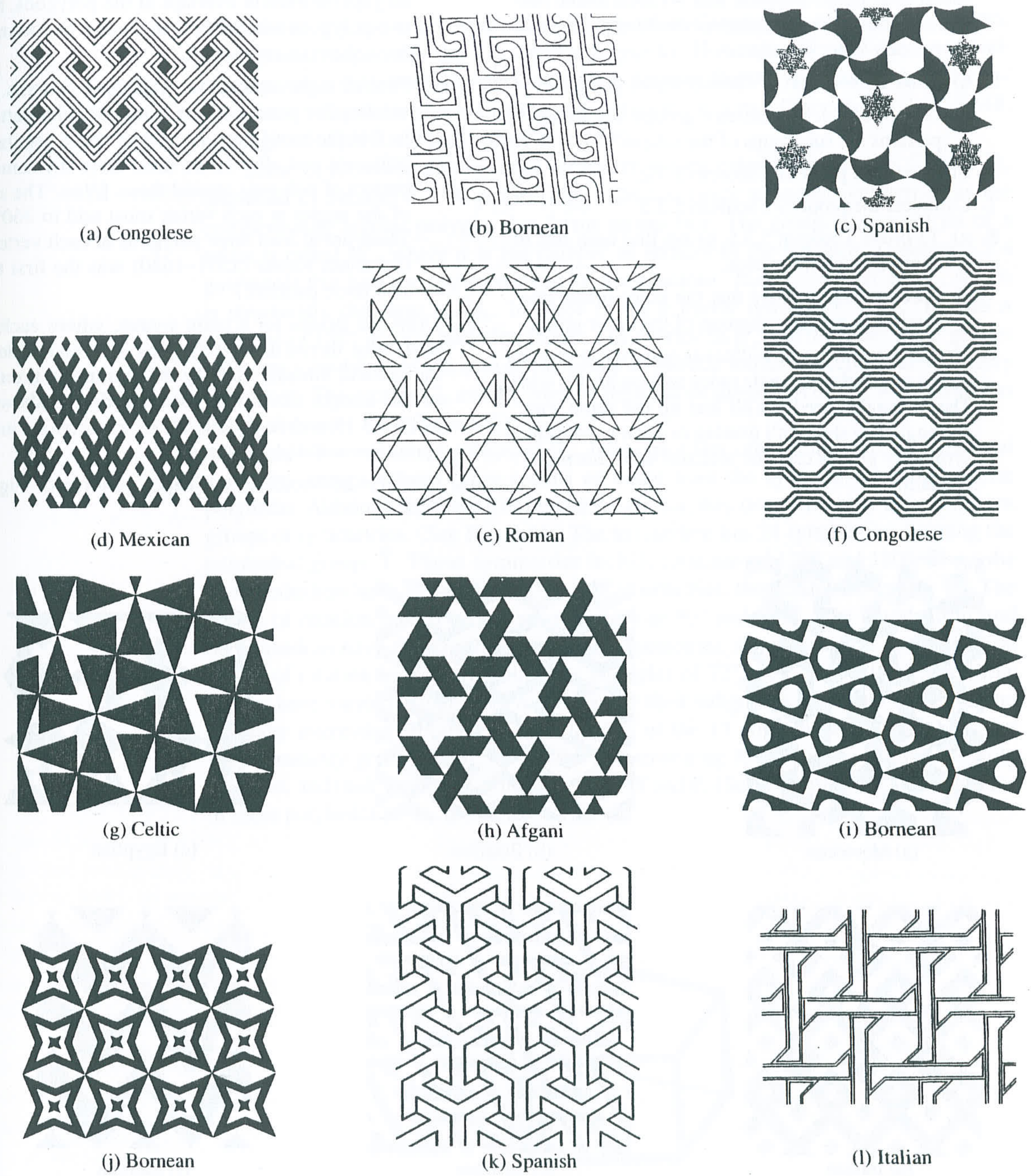


Figure 5.27

4. Prove that in every frieze pattern with horizontal translations there must be at least one stable horizontal line. [*Hint:* Suppose that no such stable line existed; show that a translation would also be possible in some other direction.

5. a) Make a flow chart to classify frieze patterns.

   b) Describe which symmetry groups for frieze patterns are subgroups of the others.

6. Complete the proof of Theorem 5.3.2.

7. Complete the proof of Theorem 5.3.3.

8. a) To prove Theorem 5.3.5, prove that both sets of symmetries form groups.

   b) (Group theory) Prove that the color-preserving group is a normal subgroup of the color group.

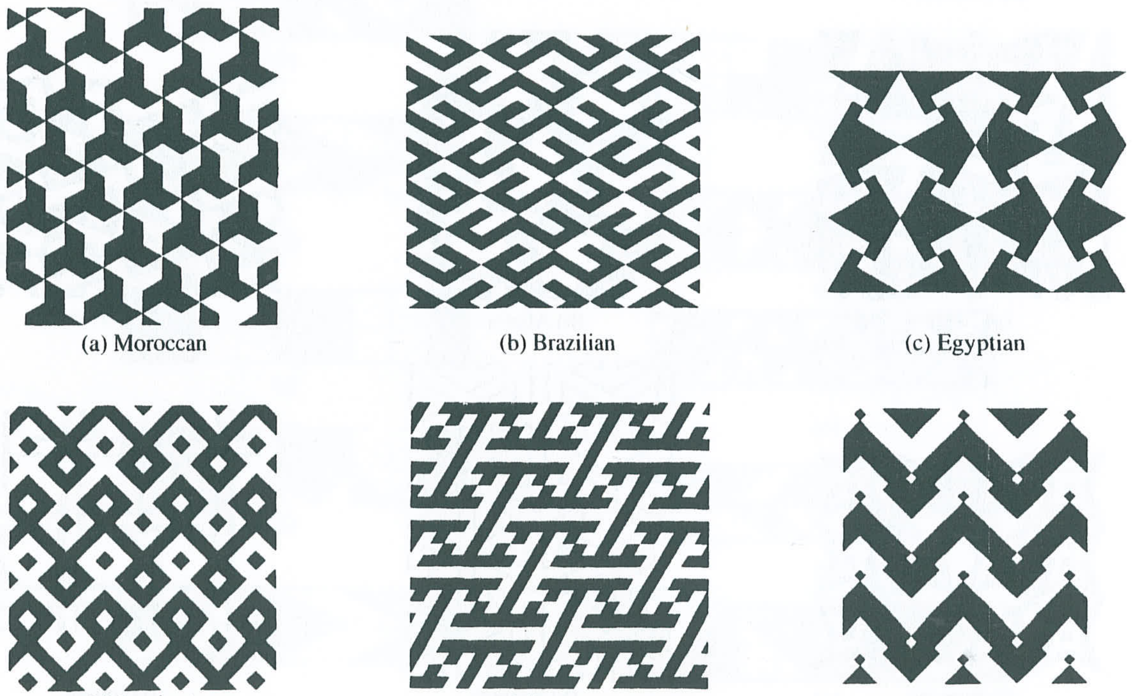
9. Find as many types of wallpaper patterns as you can with a motif of a rectangle twice as long as it is wide. The rectangles need not all line up the same way, although they shouldn't overlap or have gaps. Which symmetry groups can't be realized with this motif? Explain.

10. a) Find all regular wallpaper patterns. To be regular, the motif must be a regular polygon, there can be no gaps between or overlaps of the polygons, and two polygons with more than a point in common must share an entire edge.

    b) Find all eight semiregular wallpaper patterns. A semiregular pattern differs from a regular pattern in that the motif must be two or more regular polygons and all vertices must have the same pattern of polygons around them. [*Hint:* The sum of the angles at each vertex must add to 360°. There are at least three polygons at each vertex.] (Johannes Kepler (1571–1630) was the first to find these patterns.)

11. Draw the design for a plain weave, where each horizontal thread alternately goes over and under the vertical threads. Classify the symmetry group of this design. Classify the Spanish design shown in Fig. 5.19 and verify that all its symmetries are symmetries of a plain weave.

12. Classify the two-color wallpaper patterns of Fig. 5.28.



(a) Moroccan (b) Brazilian (c) Egyptian

(d) Iranian (e) Japanese (f) Zairean

Figure 5.28