5.2 Finite Plane Symmetry Groups

Snowflakes possess the same number of symmetries, even if no two are exactly alike. Figure 5.8 illustrates the six rotations and six mirror reflections that form the group of symmetries of a snowflake. The swastika (Fig. 5.9), a religious symbol in ancient India long before the Nazis appropriated it, has four symmetries, all rotations of multiples of 90°. In this section we classify the finite groups of plane symmetries. Leonardo da Vinci (1452–1519) realized that, in modern terms, all designs in the plane with finitely many symmetries have either rotations and mirror reflections like those in Fig. 5.8 or just rotations like those in Fig. 5.9. The two types of symmetry groups in the classification presented in Theorem 5.2.2 are called dihedral and cyclic. *Dihedral* means "two faces" and refers to the fact that the symmetries in this group can be found by using two mirrors at an angle. (See Project 1 of Chapter 4.) The argument in Theorem 5.2.2 shows how algebraic reasoning can be used to turn geometric intuition into proof.

Definition 5.2.1 The *cyclic group* $\mathbf{C}_n$ contains n rotations, all with the same center. The angles of rotation are the multiples of $360°/n$, where n is any positive integer. The *dihedral group* $\mathbf{D}_n$ contains the n rotations of $\mathbf{C}_n$ and n mirror reflections over lines passing through the center of the rotations. The angles between the lines of the mirror reflections are multiples of $180°/n$. (See Fig. 5.8.)

Exercise 1 What are the symmetry groups for Figs. 5.8 and 5.9?

Exercise 2 Find the symmetry group of a regular n-sided polygon.

Theorem 5.2.1 The isometries of a finite plane symmetry group must fix some point and so are rotations around this fixed point or are mirror reflections over lines through this fixed point.

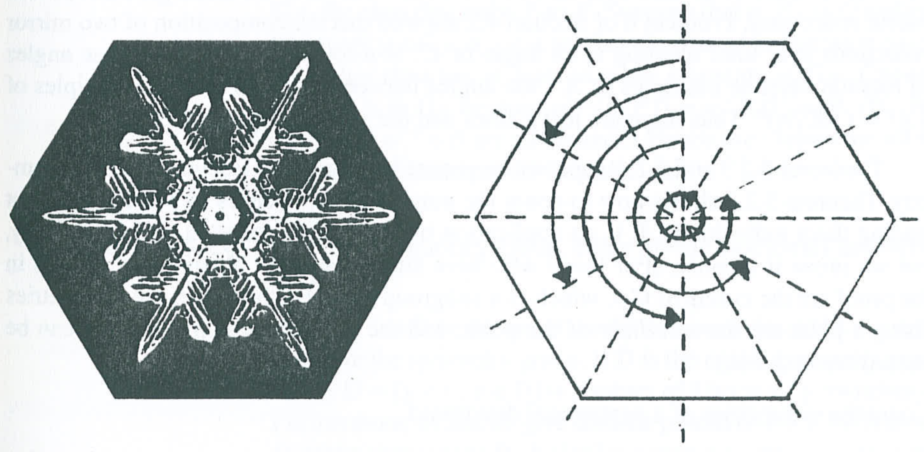


Figure 5.8 The symmetries of a snowflake.

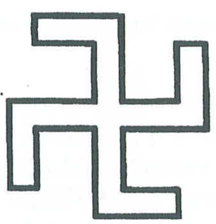


Figure 5.9

Proof. We adapt a proof from Gallian [6, 404]. Let **G** be a finite symmetry group of plane isometries and assume that the plane has coordinates. For a point A, let $S = \{\gamma(A) : \gamma \in \mathbf{G}\}$; that is, S is the set of images of A under the isometries of **G**. Because **G** is finite, so is S, whose elements can be listed as $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. The center of gravity of these n points, $(\overline{x}, \overline{y}) = (\frac{x_1+x_2+\cdots+x_n}{n}, \frac{y_1+y_2+\cdots+y_n}{n})$, must be fixed by every γ in **G**. Each (x_i, y_i) is the image of A by at least one γ_i in **G**: $(x_i, y_i) = \gamma_i(A)$. Now, for any γ in **G**, $\gamma \circ \gamma_i$ is another element of **G**, so γ will move the points of S around among themselves. Thus their center of gravity, $(\overline{x}, \overline{y})$, is fixed by each γ. From Section 4.2, we know that the only plane isometries that fix a given point are rotations around that point and mirror reflections over lines through the point. ■

Theorem 5.2.2 A finite symmetry group containing only Euclidean plane isometries is either a cyclic group or a dihedral group.

Proof. We need to show that the possible rotations and mirror reflections from Theorem 5.2.1 always fit exactly as cyclic and dihedral groups require. First, consider the rotations. If there is only one, it is the identity, a rotation of 0°. Otherwise, let the smallest positive angle of rotation be $A°$. From Chapter 4 the composition of rotations of $B°$ and $C°$ is a rotation of $B° + C°$. Thus by closure there are rotations by all multiples of $A°$. The number of rotations is finite, so A divides some multiple of 360. Moreover, A divides 360. Let kA be the largest multiple of A less than 360. Then $360 \leq (k+1)A < 360 + A$. If $(k+1)A°$ is greater than 360°, it is the same angle as $((k+1)A - 360)°$. However, this last angle would be positive and smaller than $A°$, which is a contradiction. Hence, A divides 360, say, $A = 360/n$. Thus we have at least the n rotations whose angles are multiples of $A°$.

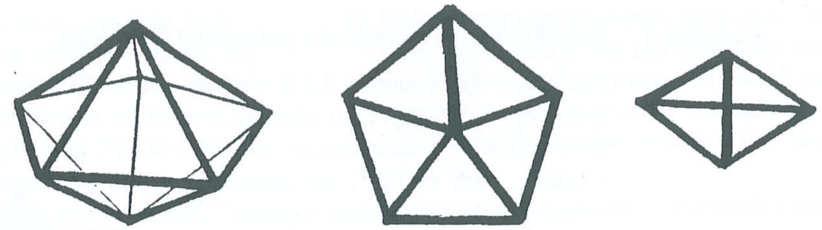
Claim. There are no others. Suppose that there were a rotation of $B°$, not a multiple of $A°$. Let jA be the largest multiple of A less than B. Then there would be a rotation of $(B - jA)°$, which would be less than $A°$, which is a contradiction. Hence the rotations form $\mathbf{C}_n$.

Next consider the mirror reflections of this symmetry group. If there are none, we have $\mathbf{C}_n$. If there is at least one, its compositions with the n rotations give n different mirror reflections. Problem 6 of Section 4.2 showed that the composition of two mirror reflections over lines meeting at an angle of $C°$ is a rotation of $2C°$. As these angles of rotation must be multiples of $A°$, the angles between the lines must be multiples of $\frac{1}{2}A° = (180/n)°$. Thus there are just n lines and the symmetry group is $\mathbf{D}_n$. ■

Theorems 5.2.3 and 5.2.4 apply more generally than just to Euclidean plane geometry. Theorem 5.2.3 shows how to count the number of symmetries of a design without finding them individually. It is an application of LaGrange's theorem in group theory, but we prove it directly. (For those who have studied abstract algebra, the classes in the proof are the cosets of $\mathbf{G}_P$, which is a subgroup of **G**. Furthermore, the symmetries fixing a point are the *stabilizer* of the point, and the points to which that point can be moved are its *orbit*.)

Example 1 Count the symmetries of a pentagonal dipyramid.

Solution. The pentagonal dipyramid shown in Fig. 5.10(a) has seven vertices, but they can't all be mapped to one another. Figure 5.10(b) shows the polyhedron from the top vertex. The symmetries fixing the top vertex form the dihedral group $\mathbf{D}_5$, so 10 symmetries fix this point. The only other vertex symmetric to the top vertex is the bottom. Hence, when we apply Theorem 5.2.3 using the top vertex, we find a total of $10 \times 2 = 20$ symmetries. We can arrive at this same number of symmetries by using one of the other vertices. Figure 5.10(c) shows that the symmetries fixing one of these other vertices form the dihedral group $\mathbf{D}_2$, which has four elements. For five such vertices, Theorem 5.2.3 again yields $4 \times 5 = 20$ symmetries. ●



(a) (b) (c)

Figure 5.10 Three views of a pentagonal dipyramid.

Theorem 5.2.3 The number of symmetries of a figure, if finite, equals the product nk, where n is the number of symmetries of the entire figure that leave a given point fixed and k is the number of points to which that point can be moved by symmetries.

Proof. Let P be any point of the figure, **G** the symmetry group, $\mathbf{G}_P$ be the set of symmetries that fix P, and $\mathbf{G}_p$ have n elements. We collect the symmetries of **G** into disjoint classes, show the classes to be the same size, n, and count the number of classes, k. Two symmetries α and β are in the same class iff they map P to the same point Q: $\alpha(P) = \beta(P) = Q$. The number of classes, k, is the number of points to which P can be moved. To complete the proof we need to show all the classes to be the same size, n, as $\mathbf{G}_P$, the class that maps P to P. Suppose that $\alpha(P) = Q$ and consider $[\alpha]$, the class of α. For every $\gamma \in \mathbf{G}_P$, $\alpha \circ \gamma$ is another element of $[\alpha]$ because $\alpha(\gamma(P)) = \alpha(P) = Q$. Hence any other class has at least as many elements as $\mathbf{G}_P$. Conversely, for every β in $[\alpha]$, $\alpha^{-1} \circ \beta$ is in $\mathbf{G}_P$ because $\alpha^{-1}(\beta(P)) = \alpha^{-1}(Q) = P$. (By Problem 4, the symmetries $\alpha \circ \gamma$ and $\alpha^{-1} \circ \beta$ are all distinct.) Hence the classes are all the same size, n. Thus the number of symmetries in **G** is nk. ■

Theorem 5.2.4 In a finite symmetry group, either all the isometries are direct or exactly half of them are direct.

Proof. Let **D** be the set of the direct isometries and **I** the set of the indirect isometries, if any, in the symmetry group. If **D** is the entire symmetry group, we are done. If $\gamma \in \mathbf{I}$, then $\gamma\mathbf{D} = \{\gamma \circ \delta : \delta \in \mathbf{D}\}$ is a subset of **I** because γ switches orientation but δ does not. Furthermore, distinct δ give distinct products $\gamma \circ \delta$ by Problem 4. Hence **I** has at least as many elements as **D**. A similar argument with $\gamma\mathbf{I} = \{\gamma \circ \beta : \beta \in \mathbf{I}\}$ shows that **D** has at least as many elements as **I**. Hence they have the same number of elements. ■

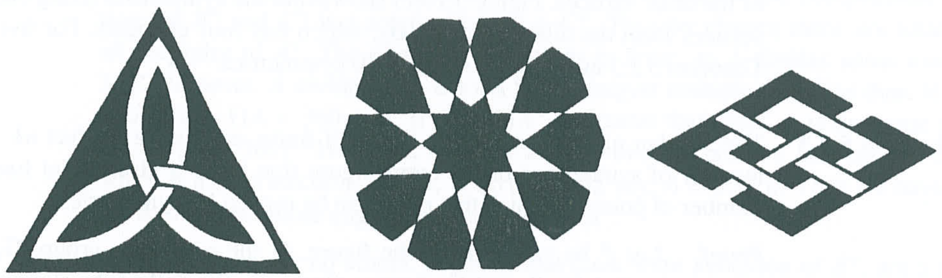
Example 2 Describe the symmetries of a pentagonal dipyramid.

Solution. By Theorem 5.2.4, we know that half of the 20 symmetries are rotations, including the identity. Four rotations around the axis through the top and bottom vertices have angles of rotation that are multiples of 72°. (See Fig. 5.10(a).) The five remaining rotations, each of 180°, are around axes that go through one of the remaining five vertices on the pentagonal "equator." There are five vertical mirror reflections and one horizontal mirror reflection. Theorem 5.2.4 guarantees four more indirect isometries, which are rotatory reflections. (See Section 4.5.) They can be written as compositions of the horizontal mirror reflection with the rotations around the vertical axis. ●

PROBLEMS FOR SECTION 5.2

1. a) Classify the symmetry group of each of the designs shown in Fig. 5.11.

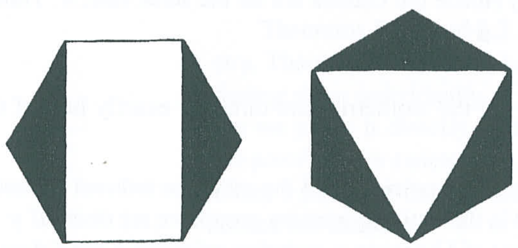
b) Classify the different types of quadrilaterals (parallelogram, kite, rhombus, and so on) by their symmetry groups.



A Gothic design An Islamic design A Gothic design

Figure 5.11

2. a) Explain why the symmetries of the rectangle shown in Fig. 5.12(a) and the symmetries of the triangle shown in Fig. 5.12(b) are symmetries of the surrounding hexagons.



(a) (b)

Figure 5.12

b) Draw an analogous design that combines two other dihedral groups.

c) Draw an analogous design that combines two cyclic groups.

d) Draw an analogous design that combines a cyclic group and a dihedral group.

e) For each of the preceding designs give the two symmetry groups. What pattern did you find between each pair of symmetry groups? Make a conjecture about these symmetry groups and try to prove it.

3. This problem introduces the idea of color symmetry. (Because this is a black and white book, we use stippling and cross-hatching to represent colors other than black and white.)

a) For the two-color (black and white) design shown in Fig. 5.13, describe the color-preserving symmetries—the symmetries that take each region to another region of the same color. Do these symmetries form a transformation group? If so, which one? Explain.

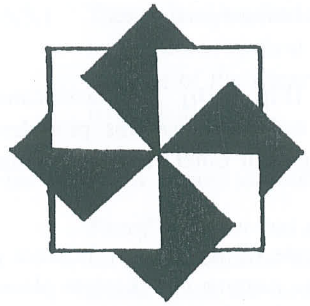


Figure 5.13

b) Describe the symmetries of this design that switch colors. Do these symmetries form a transformation group? If so, which one? Explain.

c) Call the union of the symmetries from part (a) and part (b) the *color symmetries* of the design. Do the color symmetries form a transformation group? If so, which one? Explain.

d) Make other two-color designs having different symmetries. Repeat parts (a), (b), and (c) for each of these designs.

e) Repeat parts (a), (b), and (c) for the solid black, stippled, and cross-hatched design shown in Fig. 5.14.

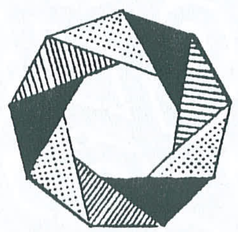


Figure 5.14

f) Make a design having at least three colors. Repeat parts (a), (b), and (c) for this design. Be sure that your color-switching symmetries take all regions of one color to the regions of a second color so that the underlying relationships of the colors are preserved.

g) Make a conjecture based on the results obtained in parts (a)–(f).

4. Show for all symmetries α, β, and γ, if $\beta \neq \gamma$, then $\alpha \circ \beta \neq \alpha \circ \gamma$ and $\beta \circ \alpha \neq \gamma \circ \alpha$. [*Hint:* Use α^{-1}.]

5. If *symmetry* is changed to *rotation* throughout Theorem 5.2.3, is the theorem still correct? If so, prove this revised theorem. If not, explain why it is false.

6. Count the symmetries of each polyhedron named.

a) The five regular polyhedra: cube, tetrahedron, octahedron, dodecahedron, and icosahedron. (See Fig. 1.44.)

b) A triangular prism and a square prism.

c) Generalize part (b) to a prism with regular n-gons for the top and bottom and n rectangles for the sides.

d) The tetrahedron has half as many symmetries as the cube. Are all the symmetries of the tetrahedron also symmetries of the cube? Explain.

7. Alter and prove Theorem 5.2.1 for isometries in three dimensions.

8. Theorem 5.2.3 can be modified to address faces rather than points. For each of the polyhedra in Problem 6, pick a face, count the symmetries taking that face to itself and the number of faces to which that face can go. Verify that the product of these numbers is the same as the number of symmetries that you found in Problem 6.

9. Modify Theorem 5.2.3 and its proof to count the symmetries of polyhedra, using faces. (See Problem 8.)

10. Relate the group of symmetries of a circle to the dihedral groups $\mathbf{D}_n$.

5.3 Symmetry in the Plane

Although real designs cannot contain infinitely many copies of a motif, many designs from various cultures convey that impression. To analyze these patterns we assume that the motif does repeat infinitely often, with translations either in just one direction