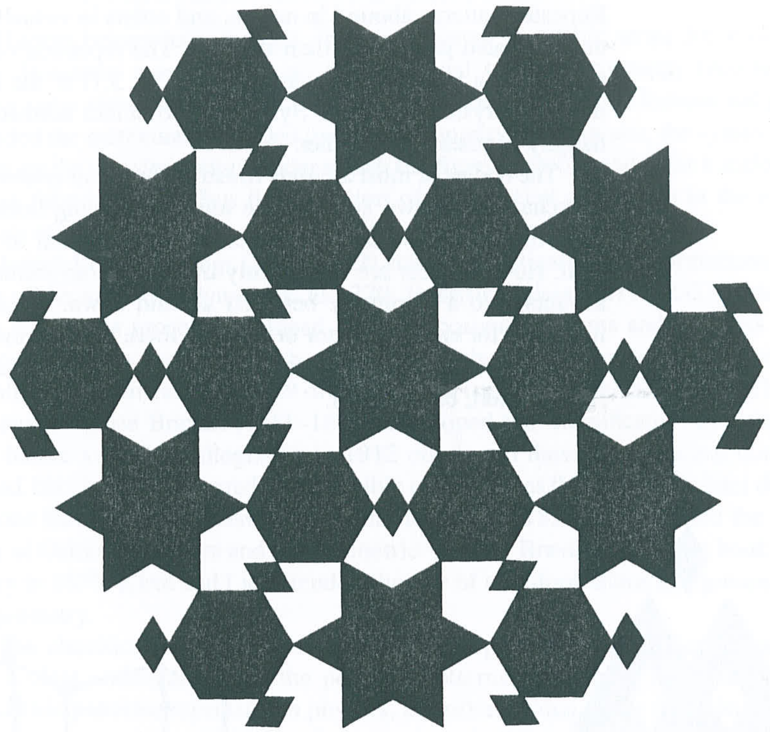
5

Symmetry



The rules of symmetry restrict how an artist can fit a repeating motif (such as this Iranian design) together to make a design. Archaeologists and anthropologists have started using symmetry in their study of designs to provide greater insight into cultures. Chemists and physicists use symmetry to organize new discoveries, to analyze empirical evidence, and to suggest fruitful lines for future inquiries. Knowing the fundamental concepts and mathematics of symmetry increases understanding in many subjects.

> Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection. —*Hermann Weyl*

> The investigation of the symmetries of a given mathematical structure has always yielded the most powerful results.
> —*Emil Artin*

5.1 Overview and History

Repeated patterns abound in nature, and artists in virtually every culture and time have used repeated patterns in their designs. The repetition of a motif underlies symmetry, whether in a bilateral Mexican design (Fig. 5.1) or the intricate atomic structure of a diamond crystal (Fig. 5.2). Symmetry combines aesthetic and practical values which happily augment one another.

The bodies of most animals illustrate *bilateral symmetry;* that is, a mirror reflection interchanges the two sides of the animal. Hunting lions as well as hunted antelopes need the ability to turn left as readily as right and to hear from each side equally well. However, feet are useful only underneath an animal, so there is no evolutionary advantage to a symmetry between up and down. Similarly, running backward isn't important for either predator or prey, so there is no symmetry between the front and the back of animals. Hence the practical needs of most animals require symmetry between right and left, but no other.

Figure 5.1 A Mexican design.

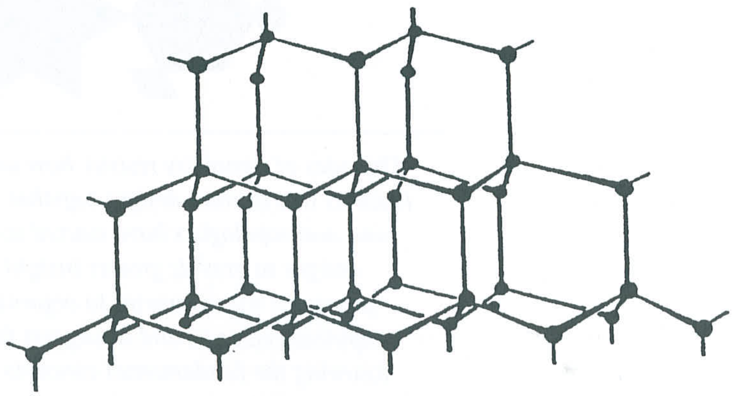


Figure 5.2 A diamond crystal.

Exercise 1 Some animals, such as jellyfish, have more than just bilateral symmetry. What makes this extra symmetry advantageous for these animals?

Evolution can explain symmetry in animals, but it provides no insight into the widespread aesthetic appreciation people have for symmetry. The unity and balance of symmetric objects seems to appeal to people of all cultures. Hermann Weyl in his classic book on symmetry [22, 3] writes:

> *Symmetric means something like well-proportioned, well-balanced, and symmetry denotes that sort of concordance of several parts by which they integrate into a whole. Beauty is bound up with symmetry.*

Human beings have thought about symmetry in artistic terms for thousands of years. However, a systematic and mathematical study of symmetry required a shift from a static viewpoint to a dynamic one. Group theory and transformational geometry provided the mathematics needed to study symmetry. In particular, the symmetries of a figure are the transformations under which the figure is stable, and these transformations always form a group. Thus the evolution of the concept of a group in the nineteenth century is inseparable from symmetry in algebra and geometry.

Joseph Louis Lagrange (1736–1813) started investigating transformations (symmetries) of the roots of polynomials in 1770. Evariste Galois (1811–1832) pursued these ideas and in the process developed many important theorems and concepts of group theory. Chemistry provided another impetus for the study of symmetry and groups, especially the classification in 1849 of all possible types of chemical crystals. The French physicist Auguste Bravais (1811–1863) developed this classification by using groups long before x-ray crystallography in 1912 confirmed these mathematical conclusions. Indeed, his classification predicted possible crystal types that were only later discovered and one that hasn't yet been found. Camille Jordan (1838–1922) united the algebraic work of Galois and others and the geometric work of Bravais in the first book on group theory in 1870. Klein and Lie extended the use of transformations and groups throughout geometry.

The classification of possible symmetry groups has supplied scientists and others with a clear understanding of the possible patterns that can be found in their areas. Symmetric patterns, especially in physics, are often formal rather than visual. Even so, the same geometric intuition underlies symmetry in that context. In turn, questions from other disciplines have stretched the notion of symmetry and raised new mathematical questions. The beauty of the mathematics of symmetry and the beauty of symmetric objects have inspired the study of symmetry.

Definition 5.1.1 A transformation σ is a *symmetry* of a subset T of a geometric space iff $\sigma(T) = T$. (Individual points in T can move to other points in T, but T is stable.) A *motif* of a design is a basic unit from which the entire design can be obtained as the images of that motif under the symmetries of the design.

The possible symmetries of a figure are usually limited to the isometries of the larger space. Problem 8 and Section 5.6 consider some other possibilities. Theorem 5.1.1 shows that the symmetries of a set form a transformation group, called the *symmetry group* of that set.

Theorem 5.1.1 The symmetries of a subset T form a transformation group.

Proof. Recall from Section 4.1 that a group of transformations needs to have closure, identity, and inverses. If α is a symmetry of T, then $\alpha(T) = T$. For closure, let α and β be symmetries of T. Then $\alpha \circ \beta(T) = \alpha(\beta(T)) = \alpha(T) = T$. Thus $\alpha \circ \beta$ is also a symmetry of T. The identity ι clearly takes any subset T to itself. Finally, let α be any symmetry of T and α^{-1} its inverse, as guaranteed by Theorem 4.1.2. Then $\alpha^{-1}(T) = \alpha^{-1}(\alpha(T)) = \iota(T) = T$, showing α^{-1} also to be a symmetry of T. ■

Example 1 The symmetry group of the design shown in Fig. 5.1 contains only the identity and a mirror reflection. Either half of the design can be considered as the motif. The symmetry group of the diamond crystal shown in Fig. 5.2 has infinitely many symmetries, if we assume that the design continues forever in all directions. For example, many translations will slide the entire crystal over to coincide with itself. Rotations of 120° around the lines representing the bonds also are symmetries. The motif in this case is one carbon atom and the bonds attaching it to its neighbors. ●

Exercise 2 Describe the symmetries of the Iranian design at the beginning of this chapter.

The variety of artistic motifs is limited only by artists' imaginations. However, these motifs can fit symmetrically in relatively few ways, which we will classify. In this chapter we discuss symmetry in two and three dimensions and present applications of symmetry in the sciences and other fields. In Section 5.6 we investigate fractals, a new area of mathematics and science that stretches the idea of symmetry in an intriguing direction. (Yaglom [23] provides further historical information; Weyl [22] is a classic study of the ideas of symmetry.)

PROBLEMS FOR SECTION 5.1

1. a) Classify the letters of the alphabet in terms of their symmetries. Consider uppercase (capital) and lowercase letters in any font you choose.

b) Find the longest word you can that has vertical bilateral symmetry (a vertical mirror reflection); repeat for horizontal bilateral symmetry. Find some words with other symmetry.

c) Make up a sentence that is a palindrome; that is, it has the same order of letters backward as forward. How does the symmetry of a palindrome differ from vertical bilateral symmetry?

2. If a figure has both horizontal and vertical mirror reflections as symmetries, must it have any other symmetry? Illustrate and explain.

3. Describe the symmetries of each design shown in Fig. 5.3.

4. Find the next few shapes in the sequence shown in Fig. 5.4.

5. a) Make two squares of the same size. Attach one corner of one of the squares to the center of the second square in such a way that the first square rotates in the plane freely around the center of the other (Fig. 5.5). Find the maximum and minimum percentages of the area of second square covered by the first square. Explain your answer. [*Hint:* Symmetry is the key.]

b) Repeat part (a) for two regular hexagons.

c) Repeat part (a) for two equilateral triangles. [*Hint:* This answer differs from the others.]

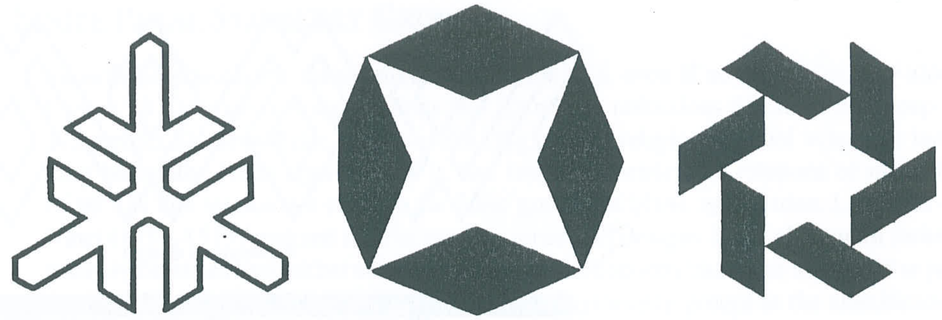
6. Let T and U be subsets of a space S. Prove that the set of transformations of S that are symmetries for both T and U form a group of transformations.

7. a) The designs shown in Fig. 5.6 are intended to continue forever in a line. Describe the types of symmetries of each design.

b) The symmetry groups for the designs shown in Fig. 5.6 differ. Create other designs having translational symmetry that have different symmetry groups. How many different types of these designs can you create?

8. a) The designs shown in Fig. 5.7 have symmetries that are affine transformations but not isometries. Describe the symmetries for these designs.

b) Create designs that have symmetries besides isometries. Identify the symmetries in each design.



An Iranian design A Byzantine design An Afgani design

Figure 5.3

Figure 5.4

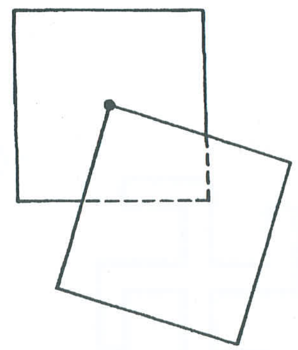
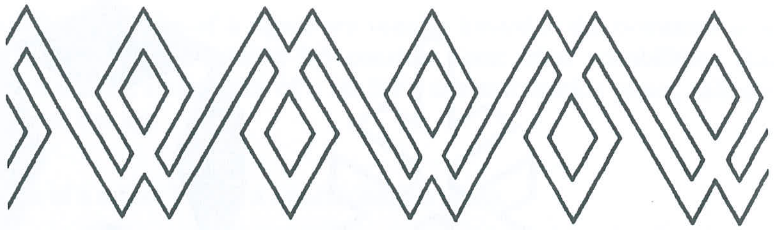
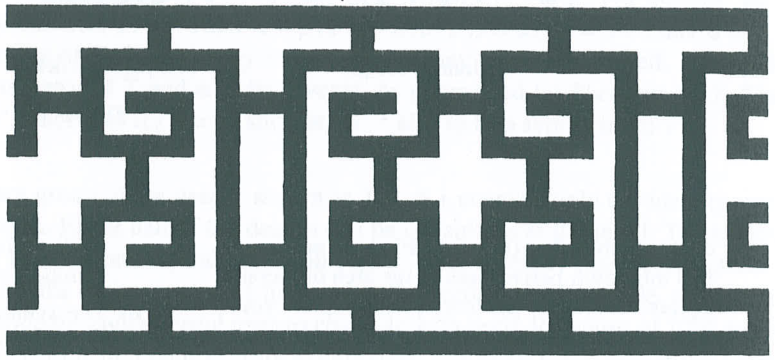


Figure 5.5



A Mexican design



A Chinese design

Figure 5.6

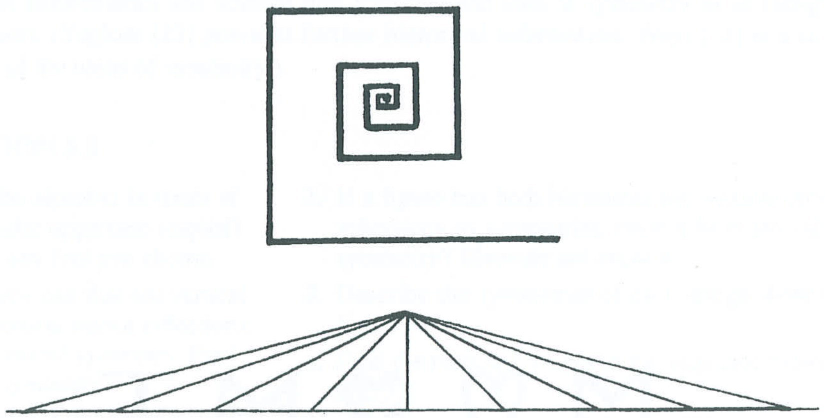


Figure 5.7