d) Find the products and fixed points of AB and AC. Describe how they differ. One of these products is again a rotation. Decide which product is a rotation and what its angle of rotation is. What can you say about the other product?

9. Recall that the standard basis of $\mathbf{R}^n$ is the set $\{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots 0, 1)\}$.

a) Prove that an $n \times n$ matrix is orthogonal iff it maps the standard basis vectors to an orthonormal basis of $\mathbf{R}^n$.

b) Explain why orthogonal $n \times n$ matrices will be isometries of the n-dimensional unit sphere.

c) Explain why the matrices we defined to be n-dimensional isometries actually are isometries.

4.6 Inversions and the Complex Plane

Affine transformations map lines to lines, but some important classes of transformations do not do so. Inversions form one important family of such "nonlinear" transformations. In brief, an inversion switches points on the inside of a circle with the points on the outside of a circle (Fig. 4.33). The center of the circle has no Euclidean point for its image, and no Euclidean point can be mapped to the center. This situation creates a problem in terms of transformations because transformations must be one-to-one onto functions. We solve this problem by adding a point to the plane that can switch places with the center. Intuitively, this extra point must be "at infinity," so we call it ∞. This new point is defined to be on every line.

Definition 4.6.1 The *inversive plane* is the Euclidean plane with one additional point, denoted ∞. Let a circle C with center O and radius r be given. The *inversion* ν_c with respect to C maps a Euclidean point P ($P \neq O$) to $\nu_c(P)$ on the ray $\overrightarrow{OP}$, where $d(O, P) \cdot d(O, \nu_c(P)) = r^2$. We define $\nu_c(O) = \infty$ and $\nu_c(\infty) = O$. The *center of the inversion* is O.

Exercise 1 Illustrate the inversion with respect to the unit circle $x^2 + y^2 = 1$. Verify that every line through the origin is mapped to itself, as is the unit circle. Explain why a circle with center $(0, 0)$ is mapped to another circle with the same center. How are the radii of these circles related?

Exercise 2 Explain why any inversion is its own inverse.

Theorem 4.6.1 Let k be any line not through the center of inversion O of ν_c. Then the image of k is a circle through O. Conversely, the image of a circle through O is a line not through O.

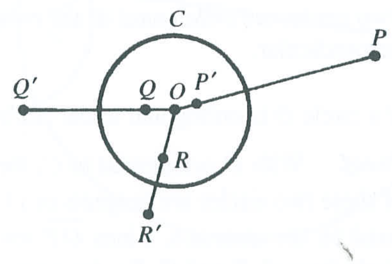


Figure 4.33 The inversion in circle C.

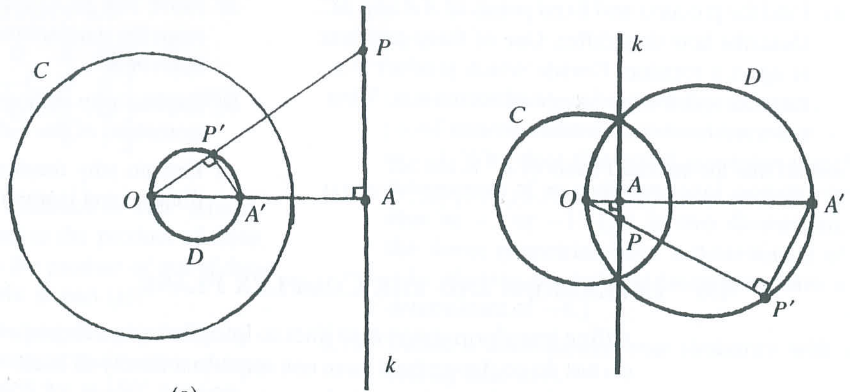


Figure 4.34

Proof. As shown in Figs. 4.34a and 4.34b, let the perpendicular from O to k intersect k at A and let $A' = \nu_c(A)$. Show that the circle D with diameter $\overline{OA'}$ is the inversive image of k. To do so, let P be any point on k and P' the (second) intersection of $\overleftrightarrow{OP}$ with the circle D. (Explain why there must be a second point of intersection.) Because $\overline{OA'}$ is a diameter, $\angle OP'A'$ is a right angle. Hence $\triangle OP'A' \sim \triangle OAP$. By the proportionality of the sides, there is some p such that $d(O, P') = p \cdot d(O, A)$ and $d(O, A') = p \cdot d(O, P)$. Then $d(O, P') \cdot d(O, P) = d(O, A) \cdot d(O, A') = r^2$ and $P' = \nu_c(P)$, as claimed. Thus D is the inversive image of k. For the other direction, note from Exercise 2 that $P' = \nu_c(P)$ iff $\nu_c(P') = P$. Hence this construction shows the inversive image of any circle D through O must be the corresponding line k. ■

Exercise 1 is a special case of Theorem 4.6.2. Another special case of that theorem, Theorem 4.6.3, is important in the Poincaré model of hyperbolic geometry. Although the proof of Theorem 4.6.2 isn't difficult, it does require lemmas from Euclidean geometry that would sidetrack us.

Theorem 4.6.2 Let D be a circle that does not pass through the center of inversion of ν_c. Then the inversive image of D is another circle that does not pass through the center of inversion.

Proof. See Eves [4, 78]. ■

Definition 4.6.2 Two circles are *orthogonal* iff the radii of these circles at their points of intersection are perpendicular.

Theorem 4.6.3 If a circle D is orthogonal to the circle of inversion C, then $\nu_c(D) = D$.

Proof. With D orthogonal to C, the radii of C that go to the intersections P and Q of these two circles are tangents to D (Fig. 4.35). Because P and Q are on C, they are fixed by the inversion. Lines $\overleftrightarrow{OP}$ and $\overleftrightarrow{OQ}$ are stable. By Theorem 4.6.2, $\nu_c(D)$ is a circle through P and Q. Furthermore, lines $\overleftrightarrow{OP}$ and $\overleftrightarrow{OQ}$ must still be tangent to $\nu_c(D)$, for each has one point of intersection with this circle. Thus the perpendiculars to these

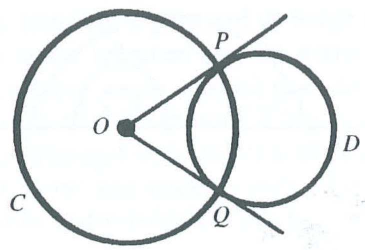


Figure 4.35

lines through P and Q intersect in the center of the circle. However, this center is also the center of D, which shows that $\nu_c(D) = D$. ■

In the Poincaré model of hyperbolic geometry, arcs of orthogonal circles are used as lines. (See Section 3.1.) Henri Poincaré had the key insight that certain inversions correspond to mirror reflections in the model now named for him. In Fig. 4.36, the points inside circle H are the points of the hyperbolic plane. Circle C is orthogonal to H, so its arc inside H is a hyperbolic line. By Theorem 4.6.3 the inversion with respect to C maps H to itself. Hence this inversion is a transformation in this model of hyperbolic geometry. For example, it switches the hyperbolic lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P'Q'}$. Using the definition of distance given in Section 3.5, Poincaré showed that this transformation is actually a mirror reflection. Euclidean mirror reflections over diameters of circle H also are hyperbolic mirror reflections. Hyperbolic mirror reflections resemble Euclidean mirror reflections in two ways. First, they switch the orientation of figures. Second, all hyperbolic plane isometries can be written as compositions of three or fewer hyperbolic mirror reflections, analogous to Theorem 4.2.5.

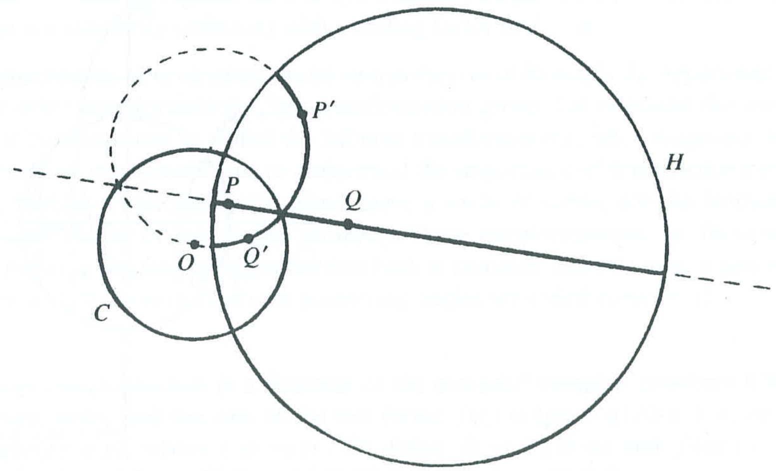


Figure 4.36 A hyperbolic mirror reflection for the Poincaré model.

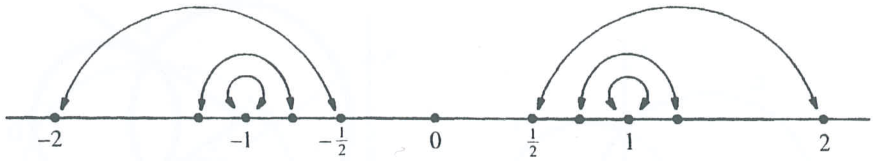


Figure 4.37

To understand hyperbolic isometries in the Poincaré model we need to consider compositions of inversions, which generally are not inversions. Complex numbers provide a convenient way to find formulas for inversions and Möbius transformations, which we discuss shortly.

Inversion is a two-dimensional analog of a function such as $f(x) = 1/x$ (Fig. 4.37), which inverts the real numbers with respect to a "circle" of radius 1 centered at 0. In effect, the real function $f(x) = 1/x$ considers just one line through the center of inversion, and 1 and -1 are the only points on the "circle." Complex numbers, of the form $a + bi$, provide a way to represent inversions on the plane. However, Example 1 shows that we need a more sophisticated function than just $1/x$ to represent inversions in the plane. Just as we needed to add a point, ∞, to the Euclidean plane to make inversions transformations, we need to extend the complex numbers. We write **C#** for the extended complex numbers, or the usual complex numbers and ∞, which is the limit of $a + bi$ as $a \to \infty$ or $b \to \infty$ or both.

Example 1 Consider the function f defined on the complex numbers by $f(z) = 1/z$. Figure 4.38 shows the images of various complex numbers. Verify that $1/(a + bi) = (a - bi)/(a^2 + b^2)$ by multiplying by $a + bi$. Verify that $f(\frac{1}{2}i) = -2i$, $f(2 - 2i) = \frac{1}{4} + \frac{1}{4}i$, and $f(-1 - 2i) = -\frac{1}{5} + \frac{2}{5}i$. Points inside the unit circle are mapped to points outside

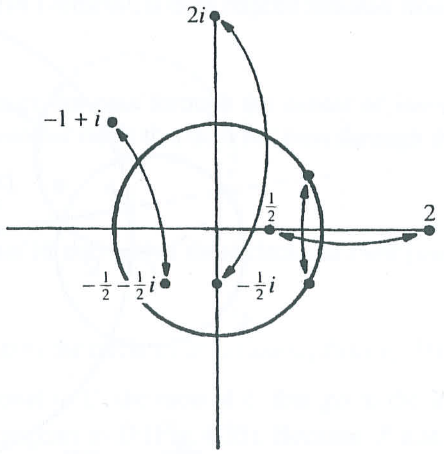


Figure 4.38 $f(z) = \frac{1}{z}$.

that circle, and vice versa. However, a point and its image are not on a line with the origin. Instead, there is a mirror reflection over the x-axis (real axis) in addition to the inversion. Fortunately, a fairly simple complex function acts as a mirror reflection over that axis: complex conjugation. Recall that $\overline{a + bi} = a - bi$. A complex number z and its conjugate $\overline{z}$ are mirror images with respect to the x-axis. Hence we can define the inversion with respect to the unit circle as $v(z) = \overline{1/z}$. Verify that $v(a + bi) = (a + bi)/(a^2 + b^2)$, a positive scalar multiple of $a + bi$. ●

Theorem 4.6.4 The inversion with respect to the circle of radius r and center w is given by $v(z) = r^2/\overline{(z - w)} + w$, for $z \neq w$.

Proof. First, show that this formula works when $w = 0 + 0i = 0$. In this case, $v_0(z)$ reduces to $r^2/\overline{z}$. The product $z \cdot r^2/\overline{z}$ has a length of r^2, showing that $r^2/\overline{z}$ is the correct distance from the origin. To complete this case, show that 0, z, and $r^2/\overline{z}$ are on the same ray. Let $z = a + bi$. Then $1/z = (a - bi)/(a^2 + b^2)$, and $r^2/\overline{z} = r^2(a + bi)/(a^2 + b^2) = r^2 z/(a^2 + b^2)$, a positive real scalar multiple of z.

We use the method in Example 3 of Section 4.3 to develop the general formula. The addition of w to every complex number is a translation τ of the complex plane. The formula $v(z) = r^2/\overline{(z - w)} + w$ is the composition $\tau \circ v_0 \circ \tau^{-1}$, which is an inversion because translations do not alter distance. To verify that this is the desired formula, let z be any point on the circle of radius r and center w. Then $z - w$ is a point on the circle of radius r at the origin. This condition implies that $r^2/\overline{(z - w)} = z - w$ from the first part. The addition of w now takes this point back to z. Thus every point on the desired circle of inversion is fixed, proving the theorem. ■

Example 2 Find the transformation of the extended complexes **C#** that is the composition of the inversions $\overline{1/z}$ followed by $\overline{4/z}$.

Solution. When we replace the z of $\overline{4/z}$ with $\overline{1/z}$, we get $4/\overline{\overline{(1/z)}} = \overline{\overline{(4z)}} = 4z$. This outcome is a similarity (dilation) with a scaling factor of 4. ●

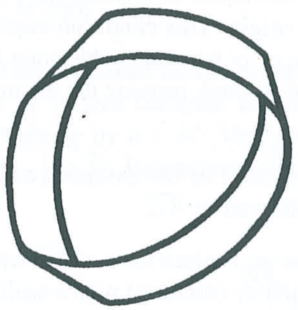
Compositions of inversions can be similarities (as in Example 2), hyperbolic isometries, or other transformations. The transformation group that contains the inversions and their compositions is called the Möbius transformations, after Augustus Möbius, one of the first mathematicians to understand the importance of transformations in geometry. Möbius transformations, which leave a circle H stable, are the isometries for the Poincaré model of hyperbolic geometry. These transformations, as Theorem 4.6.6 shows, preserve angles, an important fact both in complex analysis and in geometry. In complex analysis, transformations preserving angles are called *conformal*.

Definition 4.6.3 A *Möbius transformation* is a function of the extended complex numbers **C#** that is one-to-one, onto, and has one of the two forms $f(z) = (pz + q)/(rz + s)$ or $f(z) = \overline{(pz + q)/(rz + s)}$, where $z \neq -s/r$. We define $f(-s/r) = \infty$ and $f(\infty) = p/r$, or $\overline{p/r}$, in the second form. The constants must satisfy $ps - qr \neq 0$.

AUGUSTUS MÖBIUS

Augustus Möbius (1790–1868) earned his living as an astronomer in Leipzig, Germany, but achieved international recognition as a geometer. His absentminded concentration on mathematics often caused him to forget his keys or other things. He was very shy, which may have led him to avoid a controversy at the time between geometers about which approach, synthetic or analytic, was superior. From a modern vantage point, the quarrel seems pointless because these approaches complement one another. His work built on both approaches. In 1827 he invented barycentric coordinates, which he developed into homogeneous coordinates for projective geometry. (See Sections 2.3 and 6.3.) Julius Plucker (1801–1868) developed these coordinates into the first analytic model for projective geometry, then the cutting edge of synthetic geometry research.

Forty-five years before Klein's Erlanger Programm, Möbius investigated transformations in geometry. Even without the powerful framework of group theory, he was able to initiate the study of isometries, similarities, affine transformations, and the transformations of projective geometry. He also developed the geometry of inversions and investigated the complex transformations that now bear his name. Later in life, he initiated the study of topological transformations. At age 68, he discovered the Möbius strip, a mathematical model with the curious topological property that it has just one side and one edge as shown in the accompanying figure.



A Möbius strip.

Exercise 3 Explain why $f(-s/r) = \infty$ and why $f(z) = (pz + q)/(rz + s)$ should go to the limit p/r as z gets larger. Explain how to rewrite the inversions of Theorem 4.6.4 as Möbius transformations.

Theorem 4.6.5 The set of Möbius transformations is a transformation group.

Proof. See Problem 8. ■

Theorem 4.6.6 Möbius transformations preserve angle measure.

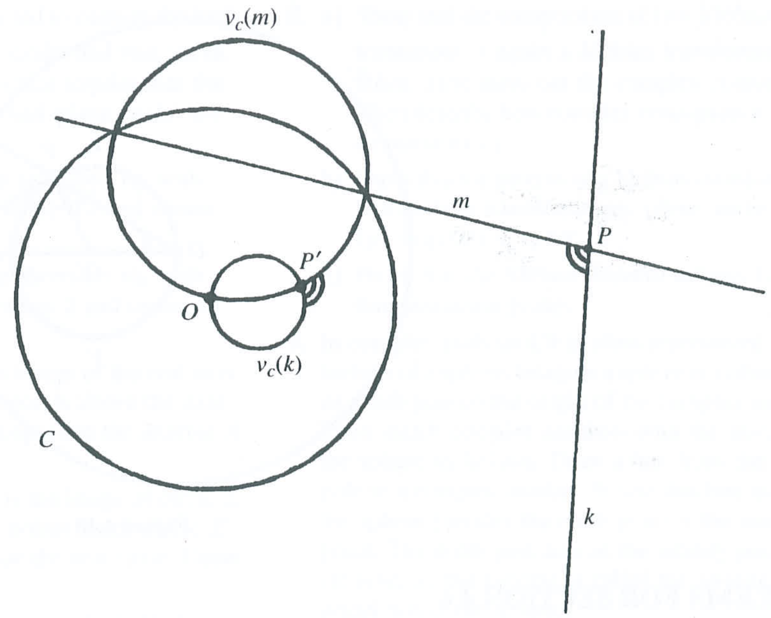


Figure 4.39

Proof. A general Möbius transformation can be written as a composition of similarities and inversions. (See Problem 6.) We already know from Theorem 4.4.3 that similarities preserve angles. Hence we only need to show that inversions preserve angles.

Recall that the angle between two curves is the angle that their tangents make at their intersection. As shown in Fig. 4.39, we know the angle between two tangents at P and are to show that the corresponding angle at P' has the same measure. We simplify this proof by comparing the angles that one of these tangents and its corresponding circle make with $\overleftrightarrow{OP}$ (Fig. 4.40). We show that $\angle PP'Q' \cong \angle OPA$ and then the angles for the other tangent and circle follow. Then we get the desired preservation of angles in Fig. 4.39 by adding the measures of these angles.

In Fig. 4.40, $\angle OP'A'$ is a right angle because $\overline{OA'}$ is a diameter, and $\angle O'P'Q'$ is a right angle because $\overleftrightarrow{P'Q'}$ is tangent to the circle and $\overline{O'P'}$ is a radius. Thus $\angle OP'O' \cong \angle A'P'Q'$. Also $\angle P'OO' \cong \angle OP'O'$ because $\triangle OO'P'$ is isosceles. Thus $\angle A'P'Q' \cong \angle P'OO'$. Angles $\angle OPA$ and $\angle P'OO'$ are complementary because they are in a right triangle. Angles $\angle PP'Q'$ and $\angle A'P'Q'$ together form a right angle, so they are complementary. Hence $\angle PP'Q' \cong \angle OPA$, as required. ■

The fundamental concept of a transformation has great significance in mathematics, linking geometry and algebra. The transformations presented are important throughout mathematics and its applications. In addition, there are many groups of transformations beyond those discussed here, including far more general ones in topology.

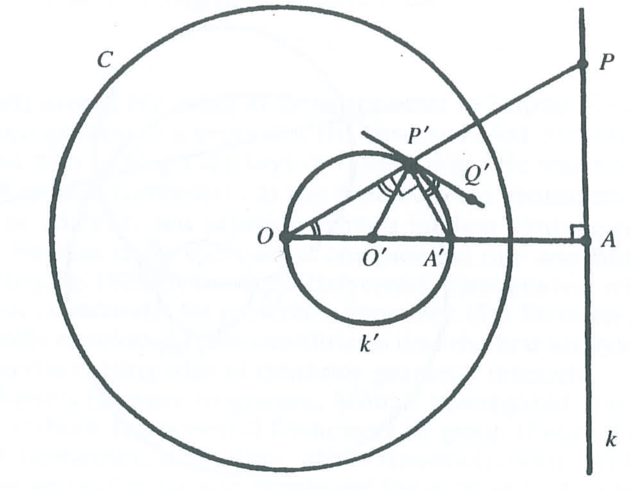


Figure 4.40

PROBLEMS FOR SECTION 4.6

1. a) Find distinct circles C and D for which $\nu_C(4, 0) = \nu_D(4, 0) = (1, 0)$.
   b) Explain why the center of any circle C or D for part (a) must be on the x-axis. Can every point on the x-axis be the center of such a circle of inversion? Explain.
   c) Find a formula for the radius of the circle from part (b) in terms of the center. [*Hint:* Look first at centers to the left of $(1, 0)$. Consider the distances from the center to $(1, 0)$ and to $(4, 0)$.]
2. a) Let P be any point outside the circle of inversion C with center O. Draw the tangents from P to C and let their points of intersection with C be Q and R. Prove that the inversive image of P is the point of intersection of $\overleftrightarrow{OP}$ and $\overleftrightarrow{QR}$. Illustrate your proof.
   b) Give a construction of the inversive image of a point that is inside the circle of inversion. Prove your construction correct.
3. In Problem 2, the line $\overleftrightarrow{QR}$ is perpendicular to $\overleftrightarrow{OP}$ and through the inversive image of P. Such a line is called the *polar* of P, and P is called the *pole* of the line $\overleftrightarrow{QR}$ with respect to the circle C. Note that the definition of a pole and a polar don't depend on the point being outside the circle. Prove that, if a point U is on the polar of a point P, then P is on the polar of U. Draw a diagram for your proof.
4. A circle D passes through a point P and the inversive image P' of P with respect to the circle C, where $P \neq P'$. Prove that D is orthogonal to C. [*Hint:* Assume that three noncollinear points determine a circle and that any path from inside a circle to outside the circle must intersect the circle.]
5. Let P', Q', and R' be the inversive images of P, Q, and R with respect to a circle C. Does Theorem 4.6.6 imply that $\triangle PQR \sim \triangle P'Q'R'$? Explore this question by using a particular circle and points. Explain your answer.
6. This problem investigates similarities as Möbius transformations.
   a) Describe the similarity $f(z) = z + (a + bi)$.
   b) If a is a nonzero real number, what similarity is $f(z) = az$? If $a + bi$ is on the unit circle, what similarity is $f(z) = (a + bi)z$? If $a + bi$ is any nonzero complex number, what similarity is $f(z) = (a + bi)z$?
   c) What similarity is $f(z) = \overline{z}$?
   d) Prove that $f(z) = (pz + q)/s$ and $f(z) = \overline{(pz + q)/s}$ are similarities for any complex numbers p, q, and s, where p and s are not 0.
   e) Prove that any Möbius transformation is the composition of a similarity and at most one inversion.

7. Möbius transformations are used to convert the half-plane model to the Poincaré model and vice versa. As Poincaré knew, this procedure implies that the hyperbolic isometries of the half-plane model are Möbius transformations.
   a) Find the function for the inversion ν_C with respect to a circle with radius $\sqrt{2}$ and center at $-2i$.
   b) Find the function for the inversion ν_D with respect to a circle with radius 2 and center at $-3i$.
   c) Find a circle E that is the image of the real axis under ν_C. Verify that the points above the axis ($a + bi$, with $b > 0$) are mapped to the interior of the circle. Draw a picture.
   d) Verify that the unit circle is the image of circle E under ν_D. Verify that the points inside circle E are mapped to points inside the unit circle. Draw a picture.
   e) The composition $\nu_D \circ \nu_C$ maps the half-plane model to the Poincaré model. For the point $(0, 1) = i$ from the half-plane model find its image in the Poincaré model.
   f) Find a composition of inversions that converts the Poincaré model to the half-plane model.
8. a) Show that the composition of two Möbius transformations is again a Möbius transformation. [*Hint:* First leave out the complex conjugates. Then describe how complex conjugates affect the compositions.]
   b) Show that the inverse of a Möbius transformation is a Möbius transformation. [*Hint:* solve $w = (pz + q)/(rz + s)$ for z.]
   c) Prove that the Möbius transformations form a transformation group.
9. In complex analysis **C**# is often represented on the surface of a sphere. Imagine a sphere of radius 1 with its south pole on the origin of the complex numbers. Then match complex numbers with the points on the sphere as follows. Draw a line from the north pole to a complex number. Where this line intersects the sphere (besides the north pole) is the matching point. The north pole acts as the infinity point ∞. (In reverse, this process is called the *stereographic projection* of the sphere.)
   a) Illustrate the process of matching complex numbers to the points on the sphere.
   b) What on the sphere corresponds to the circle $x^2 + y^2 = 4$? What spherical isometry corresponds to the inversion with respect to this circle?

PROJECTS FOR CHAPTER 4

1. Place two mirrors at an angle facing each other with a shape in their interior. Investigate the multiple images of this shape, with the use of a protractor.
   a) How do the images move as you move the original shape closer to one of the mirrors?
   b) For mirror angles of 90°, 60°, 45° and smaller, count the number of images (plus the original shape) that you can see. Find a formula relating the angle and the number of images.
   c) Use an asymmetric shape so that you can distinguish the orientation of the images. Describe the orientation of successive images of the original shape. For various mirror angles, measure as best as you can with a protractor the angle between the original shape and the first image having the same orientation. How does this angle relate to the mirror angle?
2. Place two mirrors parallel and facing each other with a shape in their interior. Investigate the multiple images of this shape.
   a) How do the images move as you move the original shape closer to one of the mirrors?
   b) Use an asymmetric shape so that you can distinguish the orientation of the images. Describe the orientation of successive images of the original shape. For various distances between the mirrors, measure as best as you can with a ruler the distance between the original shape and the first image having the same orientation. Relate the distance between the mirrors and the distance between the original and this image.
3. Place three mirrors facing each other to make three sides of a square with an asymmetric shape inside the square. Investigate the multiple images of this shape.

a) Make a diagram showing the various images and their orientations.

b) Which images have the same orientation as the original shape and which have the opposite orientation?

c) Indicate on the diagram in part (a) which isometry produces each image.

4. Use the Geometer's Sketchpad or CABRI to experiment with the effects of elementary transformations. Find the image of a triangle under various transformations and their compositions. Investigate Theorems 4.2.3–4.2.6.

5. On a transparency, draw axes and randomly insert numerous small dots (Fig. 4.41). Photocopy this transparency and align the transparency on top of the copy.

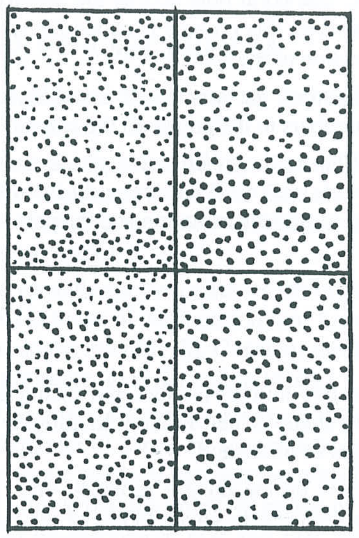


Figure 4.41

a) Rotate by a small angle the transparency relative to the paper around their common origin. Describe the pattern that the two sets of dots form. Translate the transparency relative to the paper and describe the resulting pattern. Try this procedure with other translations and small rotations. What can you say about the composition of a rotation followed by a translation?

b) Switch the order and repeat part (a). What can you say about the composition of a translation followed by a rotation? Do you get the same transformation regardless of the order?

c) Each composition in part (a) has a fixed point. Keep the initial angle of rotation the same and describe what happens to the fixed point as you use increasingly long translations in the same direction.

d) Repeat part (c), using the switched order of part (b). Compare these fixed points with those in part (c).

e) Experiment with two rotations of small angles around different points. Based on the two centers and the angles of rotation, can you predict where the new center of rotation will be?

6. Investigate geometric applications of inversions. (See Eves [4, 84–91] and Greenberg [6, 257–262].)

7. Recall that an *equivalence relation* $\equiv$ on a set S is reflexive, symmetric, and transitive. (See Gallian [5, 13–15].)

a) Suppose that **G** is a group of transformations on a set S and that, for $a, b \in S$, we define $a \equiv b$ whenever there is some $\alpha \in \mathbf{G}$ such that $\alpha(a) = b$. Prove that $\equiv$ is an equivalence relation.

b) Suppose that $\equiv$ is an equivalence relation on a set S and that we define **G** to be all transformations α on S such that, for all $a \in S$, $a \equiv \alpha(a)$. Prove that **G** is a transformation group.

c) Let S be the set of all lines in the Euclidean plane and interpret $\equiv$ as parallel. What is **G** in part (b)?

d) Let **G** be the transformation group of isometries of the Euclidean plane and S be the set of all line segments. Describe $\equiv$.

e) Repeat part (d) with S being the set of all triples of points.

f) Describe other groups of transformations and equivalence relations.

8. Investigate transformations in CAD programs. (See Mortenson [8].)

9. Investigate iterated function systems (IFSs). (See Barnsley [2] and the software, "The Desktop Fractal Design System," [6] under Suggested Media.)

10. Prove the three-dimensional analogs of Theorems 4.2.2–4.2.5

a) A three-dimensional Euclidean isometry fixing four points not all in the same plane is the identity.

b) Any three-dimensional Euclidean isometry is determined by where it maps four points not all in the same plane.

c) Given any two distinct points in three-dimensional Euclidean space, there is a unique mirror reflection switching these two points.

d) Every three-dimensional Euclidean isometry can be written as the composition of at most four mirror reflections.

11. State the analogs of Theorems 4.2.2–4.2.7 in n-dimensional Euclidean geometry.

12. Define similarities in three dimensions. State the analogs of Theorems 4.4.1–4.4.3 for three-dimensional similarities and prove them.

13. Define similarities in n-dimensions. State the analogs of Theorems 4.4.1–4.4.4 for n-dimensional similarities.

14. Investigate Theorem 4.4.6 on convex sets in n dimensions.

15. Investigate stereographic projections and other mappings of a sphere to a plane. (See Hilbert and Cohn-Vossen [7, 248–263].)

16. (Calculus) Vertical lines play a special role in calculus because functions have just one y-value for any x-value.

a) Prove that an affine matrix maps vertical lines to vertical lines iff it is of the form $\begin{bmatrix} a & 0 & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$. Prove such matrices form a transformation group.

b) Investigate what happens to the points $(x, x^2, 1)$ on the parabola $y = x^2$ under the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Graph the resulting curve. Does the minimum point on the original curve get mapped to the minimum point on the image? Find the equation of the new function h. [*Hint:* If $w = 2x + 1$, write $y = h(w)$ in terms of x and then replace x by $(w - 1)/2$. Explain why this hint works.]

c) Show that the affine matrix in part (a) transforms the function $y = g(x)$ to the function $h(ax + c) = dx + e \cdot g(x) + f$.

d) If the function g in part (c) has a derivative at every point, what can you say about the derivative of the function h? Can you find the relative maxima and minima of h? Be sure that your answers match with what you found in part (b). Experiment with other polynomials for g. What can you say about the second derivatives of g and h from part (c)?

17. Investigate hyperbolic geometry transformations. (See Greenberg [6].)

18. Investigate dynamical systems. (See Abraham and Shaw [1].)

19. Investigate topological transformations. Describe properties preserved by transformations that are more general than affine. (See Smart [9, Chapter 8].)

20. Investigate the use of transformations in biology. (See Thompson [10].)

21. Investigate the history of transformational geometry. (See Yaglom [11].)

22. Write an essay discussing Klein's definition of geometry in light of the variety in groups of transformations presented in this chapter.

Suggested Readings

[1] Abraham, R., and C. Shaw (eds.). *Dynamics—The Geometry of Behavior* (4 vols.). Santa Cruz, Calif.: Ariel Press, 1982–1988.

[2] Barnsley, M. *Fractals Everywhere*. New York: Academic Press, 1988.

[3] Coxeter, H. *Introduction to Geometry*, 2d Ed. New York: John Wiley & Sons, 1969.

[4] Eves, H. *Fundamentals of Modern Elementary Geometry*. Boston: Jones and Bartlett, 1992.

[5] Gallian, J. *Contemporary Abstract Algebra*. Lexington, Mass.: D. C. Heath, 1994.

[6] Greenberg, M. *Euclidean and Non-Euclidean Geometries*. San Francisco, Calif.: W. H. Freeman, 1980.

[7] Hilbert, D., and S. Cohn-Vossen. *Geometry and the Imagination.* New York: Chelsea, 1952.
[8] Mortenson, M. *Geometric Modeling*. New York: John Wiley & Sons, 1985.
[9] Smart, J. *Modern Geometries*, 3d Ed. Pacific Grove, Calif.: Brooks/Cole, 1988.
[10] Thompson, D. *On Growth and Form*. New York: Cambridge University Press, 1992.
[11] Yaglom, I. *Felix Klein and Sophus Lie: Evolution of the Idea of Symmetry in the Nineteenth Century*. Boston: Birkhäuser, 1988.

Suggested Media

1. "Central Similarities," 10-minute film, International Film Bureau, Chicago, 1966.
2. "Geometric Transformations," 10-minute film, Ward's Modern Learning Aids Division, Rochester, N.Y., 1969.
3. "Inversive Geometry," 24-minute video, Films for the Humanities and Sciences, Princeton, N.J., 1996.
4. "Isometries," 26-minute film, International Film Bureau, Chicago, 1967.
5. "Mr. Klein Looks at Geometry," 25-minute film, University Media, Solana Beach, Calif., 1978.
6. "The Desktop Fractal Design System," software and handbook by M. Barnsley, Academic Press, Boston, 1989.
7. "Transformations and Matrices," 25-minute film, University Media, Solana Beach, Calif., 1978.