c) The matrices A and D of this problem are two of the four matrices in Example 5. Explain how the limit set of this IFS relates to the one shown in Fig. 4.25.

12. For ease of programming, IFSs are often restricted to maps on the unit square; that is, the points $(x, y, 1)$ satisfying $0 \le x \le 1$ and $0 \le y \le 1$. Find restrictions on the coefficients of an affine matrix so that it will be a contraction mapping that sends the unit square into itself.

4.5 Transformations in Higher Dimensions; Computer-Aided Design

Transformations in three and more dimensions illustrate the power of linear algebra. We utilize the same method we used in two dimensions to move the origin: We add an extra coordinate to the vectors and represent transformations by the corresponding matrices. To generalize the definition of isometries to three and more dimensions we use the isometries of the sphere.

Interpretation By a *point* in three-dimensional affine space, we mean a column vector $(x, y, z, 1)$. By *three-dimensional affine matrix* we mean an invertible 4×4 matrix whose bottom row is $[0 \quad 0 \quad 0 \quad 1]$.

Example 1 The translation $\tau = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$ moves all points by p in the x-direction, q in the y-direction, and r in the z-direction. Note the final column gives the image of the origin $(0, 0, 0, 1)$. The upper left 3×3 submatrix describes the transformation type. ●

Exercise 1 Explain why the bottom row of an affine transformation must be $[0 \quad 0 \quad 0 \quad 1]$.

4.5.1 Isometries of the sphere

Any affine transformation that maps the unit sphere to itself necessarily maps the origin to itself. Hence spherical isometries can be represented as 3×3 matrices, in effect the upper left corner of the 4×4 affine transformations. These isometries give insights about isometries in all dimensions and the symmetries of polyhedra.

Example 2 The transformation $\rho = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is a rotation of the sphere (and all of $\mathbf{R}^3$). Points of the form $A = (a, a, a)$ are fixed by ρ and so form the axis of rotation. Furthermore, $\rho(a, b, c) = (b, c, a)$, so the composition of ρ three times will take every point back to itself, showing the angle of rotation to be 120°. In particular, the x-, y- and z-axes map to one another (Fig. 4.29). The determinant of the matrix ρ is 1, just like two-dimensional rotations. ●

Exercise 2 Verify that the transformation $\mu = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ has determinant -1 and that μ^2 is

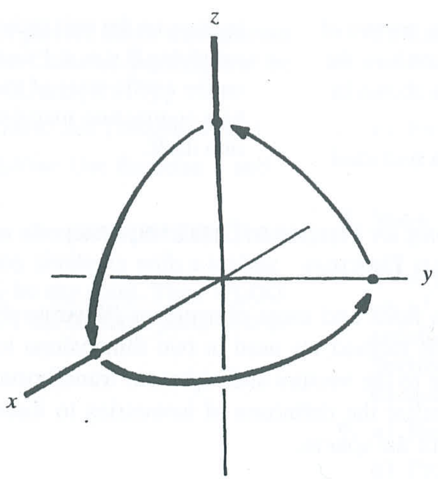

Figure 4.29 A rotation of 120° around $x = y = z$.

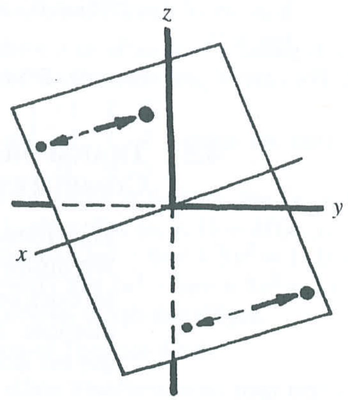

Figure 4.30 A mirror reflection in the plane $y + z = 0$.

the identity, indicating that μ is a mirror reflection. It leaves the plane $y + z = 0$ fixed (Fig. 4.30).

We can compose rotations and mirror reflections to form the only other isometries of the sphere, rotatory reflections, illustrated in Example 3. (Translations and glide reflections have no fixed points, so they aren't isometries of the sphere.)

Example 3 The rotatory reflection $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the composition of a rotation of 90° around the z-axis followed by a mirror reflection over the plane $z = 0$ (the equator) (Fig. 4.31). The eigenvalues of this matrix are -1, i, and $-i$, which show that there is no fixed point. The opposite points $(0, 0, 1)$ and $(0, 0, -1)$ are mapped to each other. ●

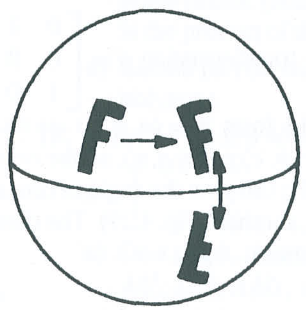

Figure 4.31 A rotary reflection composed of a 90° rotation and a mirror reflection.

Definition 4.5.1 A *rotation* in three-dimensional Euclidean geometry fixes the points on one line, called the axis of rotation, and rotates all other points through a set angle around that axis. A *mirror reflection* over a plane S in three-dimensional Euclidean geometry maps every point P to the point Q such that S is the perpendicular bisector of $\overline{PQ}$. A *rotatory reflection* in three-dimensional Euclidean geometry is the composition of a rotation with a mirror reflection in a plane perpendicular to the axis of rotation.

A spherical isometry must map the unit basis vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ to mutually perpendicular unit vectors, leading to the algebraic description of these isometries in Theorem 4.5.1. Theorem 4.5.2 gives a more geometric description of spherical isometries, although its proof depends on linear algebra.

Theorem 4.5.1 The following statements are equivalent.

i) A 3×3 matrix M is an isometry of the unit sphere.

ii) The columns of M form an orthonormal basis of $\mathbf{R}^3$.

iii) $M^{-1} = M^T$, the transpose of M.

Proof. See Problem 4. ■

Theorem 4.5.2 Every isometry of the sphere has at least two opposite points on the sphere that are either fixed or are mapped to each other. The circle on the sphere midway between these opposite points is stable.

Proof. When we use eigenvalues to find fixed points (and stable lines), we obtain an equation in λ called the *characteristic equation*. The characteristic equation of a 3×3 matrix involves a third-degree real polynomial. All odd-degree real polynomials have a real root, so every isometry of the sphere has at least one real eigenvalue. For that eigenvalue, any eigenvector of length 1 is a point on the sphere. The matrix being considered is an isometry, so the point will be mapped back onto the sphere. Thus the image also has length 1, which means that the real eigenvalue is either 1 or -1. For an eigenvalue of $+1$, the eigenvector (point) is fixed, as is its negative (opposite point). For -1, the point and its opposite change places. In either case, the circle midway between these two points must be stable because the transformation is an isometry. ■

4.5.2 Transformations in three and more dimensions

We obtain all isometries of $\mathbf{R}^3$ by combining in our general 4×4 matrix the 3×3 submatrix, representing an isometry of the sphere, and the final column of the matrix, representing a translation. In effect, we can build any isometry by composing these two special cases. In addition to the rotations, translations, mirror reflections, and glide reflections from two dimensions, there are two other types of three-dimensional isometries. Rotatory reflections, as we showed, are isometries of the sphere. Figure 4.32 illustrates the other type: screw motions.

Example 4 The matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ represents a rotation of θ around the z-axis.

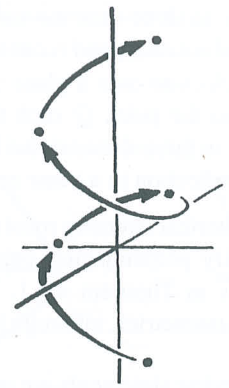

Figure 4.32 A screw motion.

A point $(0, 0, k, 1)$ on the z-axis is fixed by R. Also, each plane $z = k$ is stable under R because $R(x, y, k, 1)$ has k for its third coordinate. We define a *screw motion* as the composition of a rotation and a translation in the direction of the axis of rotation—for example, RZ, where $Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Verify that $RZ = ZR$. ●

Exercise 3 Give definitions for translations and glide reflections in three dimensions.

Theorem 4.5.3 Every three-dimensional Euclidean isometry can be written as the composition of at most four mirror reflections. There are three types of three-dimensional direct Euclidean isometries: translations, rotations, and screw motions; and three indirect types: mirror reflections, glide reflections, and rotatory reflections.

Proof. See Project 7 and Coxeter [3]. ■

In Section 4.3 we used row vectors $[a, b, c]$ to represent a line, or the set of points satisfying the equation $ax + by + c = 0$. In three dimensions $[a, b, c, d]$ represent a plane, the points satisfying $ax + by + cz + d = 0$.

By now the pattern may be clear: For points in n-dimensional affine space, use column vectors with $n + 1$ coordinates, the last of which is 1. The corresponding row vectors are, in general, called *hyperplanes* and are $(n - 1)$-dimensional. The affine transformations will be $(n + 1) \times (n + 1)$ invertible matrices whose bottom row is $[\,0 \;\; \ldots \;\; 0 \;\; 1\,]$. The upper left $n \times n$ corner tells us, up to a translation, what type of a transformation we have. Theorem 4.5.1 leads to the definition of isometries in higher dimensions.

Definition 4.5.2 An $(n + 1) \times (n + 1)$ invertible matrix is an *affine* matrix iff its bottom row is $[\,0 \;\; 0 \;\; \ldots \;\; 1\,]$. An $n \times n$ matrix M is *orthogonal* iff $M^{-1} = M^T$, where M^T is the transpose of M. An $(n + 1) \times (n + 1)$ affine matrix is an *isometry* iff its upper left $n \times n$ submatrix is orthogonal.

Exercise 4 Verify that two-dimensional isometries satisfy this definition.

Exercise 5 Define translations in n-dimensional space. What does the matrix of a translation in n dimensions look like?

4.5.3 Computer-aided design and transformations

A CAD program stores the various reference points of a design as the columns in a matrix. Matrices quickly provide the images of these points for other views of figures. The transformations of Chapter 6 enable engineers and others to give perspective views of designs by altering the bottom row of affine matrices.

Exercise 6 Draw the quadrilateral in the plane whose four corners have the columns of $A = \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 2 & 1 & -3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ for their coordinates. For the rotation $\rho = \begin{bmatrix} 0.6 & -0.8 & 1 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ of approximately 53° around $(0.5, 1, 1)$, the product $\rho A = \begin{bmatrix} 1.6 & -0.6 & -1 & 2.8 \\ 0.8 & 1.2 & -1 & -2.6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ gives the matrix whose columns are the images under ρ of the four corners of this quadrilateral. Draw the resulting quadrilateral on the same axes as the original one.

Theorem 4.5.4 Let α be an n-dimensional affine transformation and A be an $(n + 1) \times k$ matrix whose columns $A_1, A_2, \ldots, A_k$ are k points in n-dimensional affine space. Then the columns of αA are $\alpha(A_1), \alpha(A_2), \ldots, \alpha(A_k)$.

Proof. See Problem 7. ■

As a result of Theorem 4.5.4, once the computer has been given the new reference points, it can redraw the various lines, curves, and surfaces among them in the same manner as originally. The analytic geometry of Chapter 2 combined with the linear algebra of this chapter provide the graphics of CAD. Computers also use matrices to present three-dimensional designs as two-dimensional graphics displays and printouts. These matrices aren't transformations since they aren't one-to-one. (See Mortenson [8].)

Example 5 The matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ maps every point of three dimensions orthogonally onto the xy-plane. Thus two points with the same z-coordinate will be mapped to the same point. Note that the determinant of this matrix is 0. ●

PROBLEMS FOR SECTION 4.5

1. a) Explain why $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a 180° rotation around the z-axis.

   b) Find the matrices for rotations of 180° around the x- and y-axes. What is the product of these two matrices? What is the product of one of these matrices with the matrix in part (a)?

   c) Verify your answers in part (b) physically by rotating a cube 180° around the centers of two opposite faces, followed by a 180° rotation around the centers of two other opposite faces. Mark several points on the cube so that you can recognize their starting and ending positions.

   d) Repeat part (c) with 90° rotations and describe the resulting transformation.

   e) Find the matrices for the 90° rotations in part (d) and multiply them. Describe the product.

   f) Describe the matrix for a rotation of θ around the x-axis.

2. The *central symmetry* with respect to Q takes each point P to the point P', where Q is the midpoint of $\overline{PP'}$. In two dimensions, this isometry is a rotation of 180°, often called a *half-turn*.

   a) Find the matrix form of a central symmetry in three dimensions and decide what type of an isometry it is. Explain your answer.

   b) Describe the composition of two three-dimensional central symmetries.

   c) Repeat parts (a) and (b) for four and more dimensions.

3. Find the matrix for a screw motion made of a rotation of θ around the y-axis followed by a translation in the y-direction by y. Verify that you get the same screw motion if you first translate and then rotate.

4. Prove Theorem 4.5.1. [*Hint:* The ijth entry in the product AB is the inner product of the ith row of A and the jth column of B. Recall that, in an orthonormal basis, the vectors are mutually perpendicular and have length 1.

5. a) Prove that the set of spherical isometries is a transformation group.

   b) Prove that the set of orthogonal $n \times n$ matrices is a transformation group.

   c) Use the definition of an orthogonal matrix to prove that its determinant must be either $+1$ or -1. Why does this proof guarantee that the determinant of an n-dimensional isometry must also be $+1$ or -1? (As in two dimensions, the direct isometries have a determinant of $+1$, whereas the indirect isometries have a determinant of -1.)

6. a) Define a three-dimensional similarity with a scaling ratio of r.

   b) Explain why every three-dimensional similarity can be written as the product of an isometry and a dilation centered at the origin, represented by $\begin{bmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   c) What is the determinant of a three-dimensional similarity with a scaling ratio of r? What does this determinant tell you about a three-dimensional object, such as a cube, and its image under a similarity?

7. Prove Theorem 4.5.4.

8. Rotations in four dimensions extrapolate properties of rotations in two and three dimensions. For convenience, use 4×4 orthogonal matrices so that the origin $(0, 0, 0, 0)$ is fixed.

   a) Describe what is fixed by a rotation in two dimensions and by a rotation in three dimensions. What should be fixed by a rotation in four dimensions?

   b) Verify the following matrices are orthogonal with determinants of $+1$.

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

   c) The preceding matrices are rotations of the four-dimensional sphere. Find all of their fixed points ("axes") and angles of rotation.

   d) Find the products and fixed points of AB and AC. Describe how they differ. One of these products is again a rotation. Decide which product is a rotation and what its angle of rotation is. What can you say about the other product?

9. Recall that the standard basis of $\mathbf{R}^n$ is the set $\{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots 0, 1)\}$.

   a) Prove that an $n \times n$ matrix is orthogonal iff it maps the standard basis vectors to an orthonormal basis of $\mathbf{R}^n$.

   b) Explain why orthogonal $n \times n$ matrices will be isometries of the n-dimensional unit sphere.

   c) Explain why the matrices we defined to be n-dimensional isometries actually are isometries.