d) Show the set of isometries fixing a given point A to be a transformation group.

e) Show the set of isometries leaving a line stable to be a transformation group.

f) Show that the indirect isometries are not a transformation group. Which properties for a transformation group fail?

11. a) Prove that an invertible matrix M never has $\lambda = 0$ as an eigenvalue.

b) If $\lambda \neq 0$ is an eigenvalue of an affine matrix M, prove that $1/\lambda$ is an eigenvalue of M^{-1}.

4.4 SIMILARITIES AND AFFINE TRANSFORMATIONS

4.4.1 Similarities

Intermediate between isometries and affine transformations are *similarities* (also called *similitudes*), the transformations corresponding to similar figures. (See Section 1.5.)

Definition 4.4.1 A transformation σ of the Euclidean plane is a *similarity* iff there is a positive real number r such that for all points P and Q in the plane, $d(\sigma(P), \sigma(Q)) = r \cdot d(P, Q)$. The number r is the *scaling ratio* of σ.

Example 1 In Fig. 4.20, we can transform the smaller triangle into the larger similar triangle by rotating it 90°, translating it, and then using a scaling ratio of $r = 1.5$. ●

Theorem 4.4.1 The set of similarities forms a transformation group.

Proof. See Problem 2. ■

Example 2 The matrix $\begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}$ takes $(x, y, 1)$ to $(rx, ry, 1)$. Thus the origin $(0, 0, 1)$ is fixed, and all points expand or contract with respect to the origin by a scaling ratio of r. ●

Theorem 4.4.2 An affine matrix M represents a similarity iff

$$M = \begin{bmatrix} r\cos\theta & \mp r\sin\theta & a \\ r\sin\theta & \pm r\cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}, \text{ for some } r > 0.$$

Proof. See Problem 4. ■

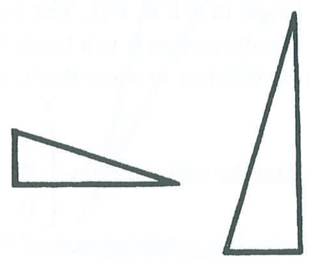

Figure 4.20

Exercise 1 Show that the matrix form of a similarity can be written as the product of an isometry and the matrix given in Example 2.

Theorem 4.4.3 Similarities preserve angle measures and the proportions of all distances. If a similarity has a scaling ratio of r, then the area of the image of a convex polygon is r^2 times the area of the original polygon.

Proof. See Problem 5. ■

Theorem 4.4.4 A similarity with a scaling ratio of $r \neq 1$ has a unique fixed point.

Proof. Let M be the similarity matrix and $(u, v, 1)$ be a candidate for a fixed point. Theorem 4.4.2 gives one of the following systems of two equations in the two unknowns u and v:

$$\left\{\begin{array}{ll} r\cos(A)u - r\sin(A)v + a & = u \\ r\sin(A)u + r\cos(A)v + b & = v \end{array}\right\} \text{ and } \left\{\begin{array}{ll} r\cos(A)u + r\sin(A)v + a & = u \\ r\sin(A)u - r\cos(A)v + b & = v \end{array}\right\}.$$

The first system becomes $\left\{\begin{array}{ll} (r\cos A - 1)u - r\sin(A)v & = -a \\ r\sin(A)u + (r\cos A - 1)v & = -b \end{array}\right\}$, which has a unique solution when the determinant isn't zero. The determinant is $(r\cos A - 1)^2 + r^2\sin^2 A = r^2 - 2r\cos A + 1$. By the quadratic formula, $r^2 - 2r\cos A + 1 = 0$ only if $r = (2\cos A \pm \sqrt{4\cos^2 A - 4})/2$. The value under the $\sqrt{\ }$ is negative except when $\cos^2 A = 1$, which forces $r = \pm 1$. As $r > 0$, the determinant can be 0 only when $r = 1$. Hence, if $r \neq 1$, there must be a unique fixed point.

Similarly, the second system reduces to a system whose determinant is $1 - r^2$. Again, for $r > 0$ and $r \neq 1$, we have a nonzero determinant and so a unique fixed point. ■

4.4.2 Affine transformations

Theorem 4.4.5 The set of affine transformations is a transformation group. Affine transformations preserve lines, parallelism, betweenness, and proportions on a line.

Proof. We prove betweenness and proportions on a line. (See Problem 9 for the rest.) From Section 2.5, a point R (considered as a vector) is on line $l = \overleftrightarrow{PQ}$ iff $R = P + r(Q - P)$, where r is a real number. (For example, $r = 0$ gives P and $r = 1$ gives Q.) In effect, the values of r give the coordinates for the points on line l, and $|r|$ is the ratio of the length of $\overline{PR}$ to $\overline{PQ}$. The point R is between P and Q iff $0 \leq r \leq 1$. Note that any affine transformation α is a linear transformation, so $\alpha(R) = \alpha(P) + r(\alpha(Q) - \alpha(P))$, which is sufficient to show that α preserves betweenness and proportions on a line. ■

Example 3 Verify that the affine matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ triples every x-coordinate, doubles every y-coordinate, has a determinant of 6, and increases the area of every triangle by a factor of 6.

SOPHUS LIE

The Norwegian Sophus Lie (1842–1899) achieved international status in mathematics for his profound work in continuous transformation groups. Lie met Felix Klein in Berlin, and they quickly became close friends. Together they went to Paris in 1870 and studied groups, but the outbreak of the Franco–Prussian War in 1871 ended their studies. Lie decided to spend his enforced vacation hiking in the Alps. As a tall, blond stranger with poor French he was soon arrested as a spy. He spent a month in prison, working on mathematics problems. When he was freed, in part owing to a French mathematician's efforts, he continued his hiking tour.

Both Lie and Klein were deeply influenced by the possibility that group theory could unify mathematical thinking. Lie first applied transformation groups in the field of differential equations, classifying solutions. He searched more broadly for an understanding of all continuous transformation groups, a goal that hasn't yet been achieved. He pioneered the study of these groups, now called Lie groups. He revealed the geometric structure of these groups and went on to develop algebras, also named for him, that matched these groups. Lie groups and Lie algebras are essential ideas in quantum mechanics, a part of physics, as well as in mathematics.

Lie applied his profound understanding of transformations to solve in 1893 a geometry problem posed by Hermann von Helmholtz. Helmholtz sought to use curvature to characterize all continuous homogeneous geometries, that is, geometries in which rigid bodies could be freely moved. Lie used transformations to prove that the only such spaces were Euclidean, hyperbolic, spherical, and single elliptic geometries, in any number of dimensions.

Solution. We leave all but the last part to you. For the last part, note that every triangle without a horizontal side can be split into two triangles, each with a horizontal side (Fig. 4.21). Then each triangle has its base tripled and its height doubled. Hence its area is multiplied by 6. As this example suggests, the absolute value of the determinant of an affine matrix is the scaling factor for areas, as is the case for similarities. ●

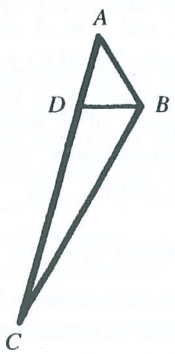

Figure 4.21

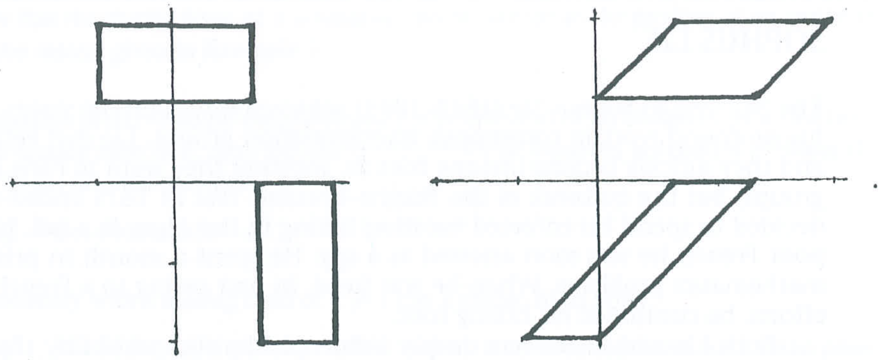

Figure 4.22

Example 4 The affine matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an example of a *shear* transformation, where all points move in parallel, but have different displacements (Fig. 4.22). Note that the determinant of this matrix is 1, preserving area. Geologic shears display the same effect on parts of the earth's crust as geometric shears do on the plane.

Shears and other affine transformations can distort shapes—for example, altering a circle to become an ellipse. Theorem 4.4.6 gives one limit to the amount of distortion of affine transformations: A convex set is always mapped to a convex set. (The converse holds as well: Transformations of the Euclidean plane that preserve convexity are affine transformations.) ●

Theorem 4.4.6 An affine transformation preserves convexity.

Proof. Suppose that α is any affine transformation and that A is any convex set. We must show that $\alpha(A)$ also is convex. That is, for X' and Y' any two points in $\alpha(A)$ and Z', any point between them, we must show that Z' is in $\alpha(A)$. By definition of $\alpha(A)$, there are X and Y in A such that $\alpha(X) = X'$ and $\alpha(Y) = Y'$. Furthermore, there is a unique Z such that $\alpha(Z) = Z'$. We need only show that Z is between X and Y because then the convexity of A will guarantee that Z is in A and so Z' is in $\alpha(A)$. Consider α^{-1}, the inverse of α, which is also an affine transformation. By Theorem 4.4.5, α^{-1} preserves betweenness. As Z' is between X' and Y', Z is between X and Y, implying that $\alpha(A)$ is convex. ■

4.4.3 Iterated function systems

Whereas the affine image of a convex set is always convex, iterated function systems (abbreviated IFSs) combine affine transformations in an unusual way to yield highly nonconvex sets called *fractals*. The points in a fractal are all the limits of infinitely many applications of the various affine transformations in all possible orders. The figure

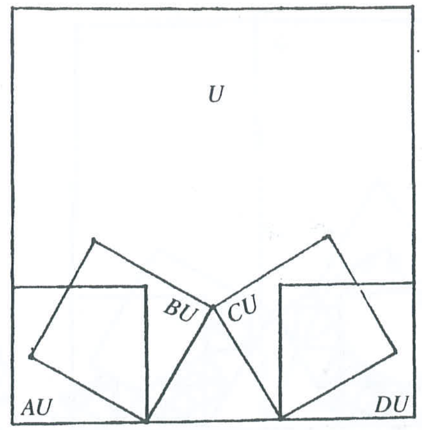

Figure 4.23

introducing this chapter shows such a fractal. A simpler example reveals the idea behind this approach to fractals introduced by Michael Barnsley.

Example 5 Matrices A, B, C, and D shrink the unit square U shown in Fig. 4.23 to the four smaller squares shown. $A = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \cos 60°/3 & -\sin 60°/3 & 1/3 \\ \sin 60°/3 & \cos 60°/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} \cos 300°/3 & -\sin 300°/3 & 1/2 \\ \sin 300°/3 & \cos 300°/3 & \sqrt{3}/6 \\ 0 & 0 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus any starting point P in U must map to some point P' in one of the small squares in Fig. 4.23. In turn, P' is mapped to some point P'' in the 16 even smaller squares in Fig. 4.24. If we continue this process infinitely, any point will end up somewhere along the extremely convoluted Koch curve shown in Fig. 4.25, which fits into all the nested squares. ●

For the practical goal of drawing interesting pictures, we don't need to iterate the process of Example 5 infinitely many times. Even six iterations will shrink the original square to a multitude of squares so small that they look like points. Each iteration quadruples the number of squares and shrinks the sides of each to one-third its previous size. Thus, after six iterations, there will be $4^6 = 4096$ squares, each with a side of length $1/3^6 = 1/729$. Barnsley realized that it is faster still to apply a random sequence of the four defining transformations to one point and plot the resulting images. These images quickly approximate the theoretical curve as closely as the eye can see.

Not every affine matrix can be part of an IFS, for the matrices must contract all distances to ensure limit points to form the fractal. Similarities with a scaling ratio $r < 1$ contract distances, as do some other affine matrices.

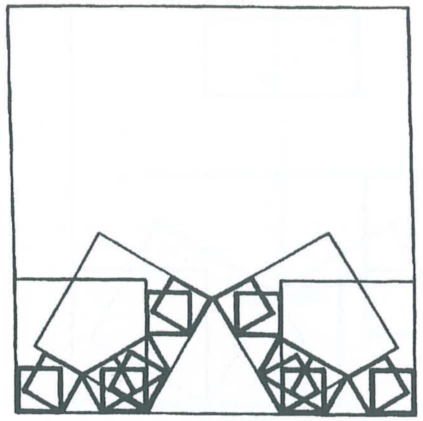

Figure 4.24

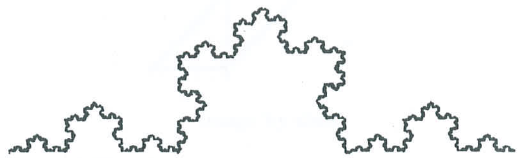

Figure 4.25 The Koch curve.

Example 6 Figure 4.26 shows a distorted Koch curve made from the four matrices

$$A' = \begin{bmatrix} 1/3 & 1/6 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B' = \begin{bmatrix} \cos 60°/3 & \cos 60°/6 - \sin 60°/3 & 1/3 \\ \sin 60°/3 & \sin 60°/6 + \cos 60°/3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C' = \begin{bmatrix} \cos 300°/3 & -\cos 300°/6 - \sin 300°/3 & 1/2 \\ \sin 300°/3 & -\sin 300°/6 + \cos 300°/3 & \sqrt{3}/6 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$D' = \begin{bmatrix} 1/3 & -1/6 & 2/3 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Figure 4.27 shows the images of the unit square for each of these matrices, which are products of the shears $\begin{bmatrix} 1 & \pm 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with the matrices of Example 5. Shears shift rectangles to parallelograms, as shown in Fig. 4.22. ●

Definition 4.4.2 An affine transformation κ is a *contraction mapping* iff there is a real number r with $0 < r < 1$ such that for all points P and Q, $d(\kappa(P), \kappa(Q)) \leq r \cdot d(P, Q)$. An *iterated function system* (IFS) is a finite set of contraction mappings. The set of limit points resulting from infinitely many applications of the contraction mappings in all possible orders is an *IFS fractal*.

Theorem 4.4.7 Every contraction mapping has a unique fixed point. An IFS fractal is a closed and bounded set.

Proof. See Barnsley [2] for a proof. The reasoning of Problem 6(a) applies to any contraction mapping to give a unique fixed point. Note that in Example 4 the fractal is bounded by the sequence of squares. ■

Exercise 2 Explain why the contraction mappings do not form a transformation group.

Figure 4.26 A distorted Koch curve.

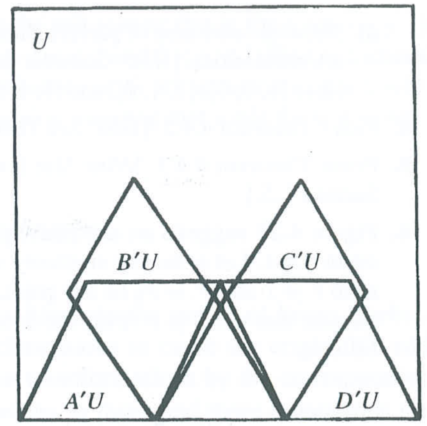

Figure 4.27

Figures 4.25 and 4.26 and the fractal introducing this chapter were made by using two-place decimals for each matrix entry. In other words only 12 digits per matrix times the number of matrices are needed to encode any such curve. Five matrices, and so 60 digits, are sufficient to produce the intricacies of the fractal introducing this chapter. Barnsley has patented a way to replace an entire picture of a real scene with a collection of IFSs. This method allows him to compress pictorial information into a small set of numbers. A computer program can quickly recover the picture from the set of numbers. Currently, one CD-ROM encyclopedia encodes its 10,000 pictures as IFS. (See Barnsley [2] for a detailed development of IFSs.)

PROBLEMS FOR SECTION 4.4

1. a) Let $M = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $X = (1, 0, 1)$. Find and graph the points X, MX, M^2X, and M^3X. Describe the shape of the curve that appears to go through these points.

b) You can fill in some of the points on the curve of part (a) by finding a matrix S that is the "square root" of M. Verify that $S = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ satisfies $S^2 = M$. Find and graph on the same axis as part (a) the points SX, S^3X, and S^5X. Do they fill in the curve you described in part (a)?

c) For $M = \begin{bmatrix} r\cos\theta & -r\sin\theta & 0 \\ r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$$S = \begin{bmatrix} \sqrt{r}\cos(\theta/2) & -\sqrt{r}\sin(\theta/2) & 0 \\ \sqrt{r}\sin(\theta/2) & \sqrt{r}\cos(\theta/2) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

use trigonometry to prove that $S^2 = M$. Explain how S and M move points.

d) For M in part (c), find a "cube root" C and an "nth root" N and explain why they qualify as a cube root and an nth root of M.

2. Prove Theorem 4.4.1.

3. An affine transformation δ is a *dilation* (or *dilatation*) provided that, for every line k, $k \| \delta(k)$.

a) Prove that the set of dilations form a transformation group.

b) Show that lines parallel to $[s, t, u]$ have the form $[s, t, w]$ or $[rs, rt, rw]$, for $r \neq 0$.

c) Show that, if $M = \begin{bmatrix} x & 0 & c \\ 0 & x & f \\ 0 & 0 & 1 \end{bmatrix}$, where $x \neq 0$, then M is a dilation. Describe how M moves points for various x.

d) Show the converse of part (c). Show that dilations are similarities. [*Hint:* Consider the images of the lines [1, 0, 0], [0, 1, 0], and [1, 1, 0].]

4. Prove Theorem 4.4.2. [*Hint:* See Theorem 4.3.2.]

5. Prove Theorem 4.4.3. [*Hint:* Use Exercise 1 and Section 1.5.]

6. Figure 4.28 suggests an alternative proof for Theorem 4.4.4. Let α be any similarity with a scaling ratio $r \neq 1$ and $P = P_0$ be any point. Then WLOG assume that $\alpha(P) \neq P$. Find the fixed point of α as follows.

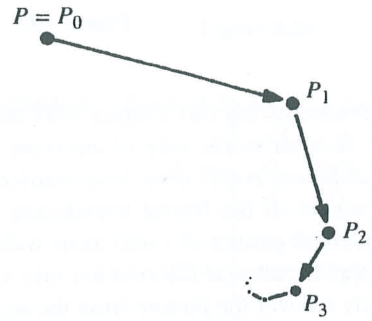

Figure 4.28

a) (Calculus) Case 1: $r < 1$. Let $P_1 = \alpha(P) = \alpha(P_0)$ and, in general, $P_{n+1} = \alpha(P_n)$. What happens to $d(P_{n+1}, P_n)$ as $n \to \infty$? Explain why there must be a point $Q = \lim_{n\to\infty} P_n$. Explain why Q must be a fixed point of α. (A proof of this fact uses analysis and so is beyond the level of this book.)

b) Case 2: $r > 1$. Show that the scaling ratio for α^{-1} is $1/r < 1$. By part (a), α^{-1} has a fixed point. Prove that α has the same fixed points as α^{-1}.

c) Prove that, if $r \neq 1$, there cannot be two distinct fixed points of α.

7. Show that any triangle ΔABC can be transformed to any other triangle ΔPQR by some affine transformation.

a) Give general coordinates for P, Q, and R. Find the matrix M that takes $O = (0, 0, 1)$ to P, $X = (1, 0, 1)$ to Q, and $Y = (0, 1, 1)$ to R. Explain why M is a transformation provided that P, Q, and R aren't collinear.

b) Let A, B, and C be any three noncollinear points. Show that there is an affine matrix N that takes A to O, B to X, and C to Y.

c) Prove that there is an affine matrix taking A to P, B to Q, and C to R.

8. Show that $M = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ maps the circle $x^2 + y^2 = 1$ to an ellipse as follows. For $(x, y, 1)$ on $x^2 + y^2 = 1$, show that $(u, v, 1) = M(x, y, 1)$ satisfies the equation $2u^2 - 6uv + 5v^2 = 1$. Use Section 2.2 to verify that $2u^2 - 6uv + 5v^2 = 1$ is the equation of an ellipse. Graph this ellipse.

9. a) Prove the rest of Theorem 4.4.5.

b) Prove that affine transformations map rays to rays.

10. Prove that an affine matrix with determinant d changes the area of convex polygons by a factor of $|d|$, as follows.

a) Show that a matrix $A = \begin{bmatrix} 1 & p & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{bmatrix}$ changes the area of any triangle by a factor of $|q|$. [*Hint:* Show that A takes horizontal lines to horizontal lines and use Exercise 2.]

b) Show a claim similar to part (a) for a matrix $B = \begin{bmatrix} r & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

c) Show that a general affine matrix M can be written as the product TAB (or TBA if the center entry of M is 0), where T is a translation, A is a matrix of the form in part (a), and B is a matrix of the form in part (b).

d) Use part (c) to extend parts (a) and (b) to any affine matrix. [*Hint:* The determinant of a product is the product of the determinants.]

e) Extend part (d) from triangles to convex polygons.

11. Consider the IFS with just two matrices,
$$A = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

a) On a graph, show the unit square U and its images AU, DU, AAU, ADU, DAU, and DDU.

b) Describe and, as best as you can, draw the fractal of this IFS. (It is a very disconnected set called the Cantor set.)

c) The matrices A and D of this problem are two of the four matrices in Example 5. Explain how the limit set of this IFS relates to the one shown in Fig. 4.25.

12. For ease of programming, IFSs are often restricted to maps on the unit square; that is, the points $(x, y, 1)$ satisfying $0 \le x \le 1$ and $0 \le y \le 1$. Find restrictions on the coefficients of an affine matrix so that it will be a contraction mapping that sends the unit square into itself.