angles are Euclidean properties, as is the shape of a figure. However, the orientation of figures isn't a Euclidean property because mirror reflections and glide reflections switch orientation. Also, verticality isn't a Euclidean property because some isometries, such as a rotation of 45°, tilt vertical lines. If we wanted to study orientation or verticality, we would need to use different groups of transformations, and, according to Klein, we would be studying a different geometry.

PROBLEMS FOR SECTION 4.2

1. Suppose that an isometry β takes $(1, 0)$ to $(-1, 0)$, $(2, 0)$ to $(-1, -1)$ and $(0, 2)$ to $(1, 1)$, respectively. Find the images of $(0, 0)$ and $(2, 2)$ and of a general point (x, y). Draw a figure showing these points and their images.

2. Outline the original placement of a small rectangular piece of paper on a larger piece of paper. Label the corners of both the small rectangle and the outline A, B, C, and D so that you can determine the rectangle's movements. Note that the centers of rotation are on the outline and do not move.

a) Rotate the small rectangle 180° around A and then 180° around C on the outline. Describe the resulting transformation.

b) Return the small piece of paper to its starting position and repeat part (a) but switch the order of the rotations. Describe how this new transformation differs from the one in part (a).

c) Repeat parts (a) and (b) but use rotations of 90° at A and C.

d) Repeat part (c) but rotate the rectangle 90° around A followed by a rotation of −90° around C.

e) Repeat part (c) with various angles and centers of rotations. Make a conjecture about the resulting transformations.

3. a) If μ_k is a mirror reflection over the line k and τ is a translation in the direction of k, investigate whether $\mu_k \circ \tau = \tau \circ \mu_k$ and justify your answer. [*Hint:* It may help to do this first physically with a triangle placed on a sheet of paper. Draw the line k on the paper. Geometer's Sketchpad or CABRI also will help.]

b) Find three mirror reflections whose composition is a glide reflection.

c) What is the composition of a glide reflection with itself? Justify your answer.

4. a) If α is an isometry which fixes two points, prove that α is the identity or the mirror reflection over the line through the fixed points.

b) If α and β are isometries such that $\alpha(A) = \beta(A)$ and $\alpha(B) = \beta(B)$, prove that $\alpha = \beta$ or $\alpha = \beta \circ \mu$, where μ is the mirror reflection over the line $\overleftrightarrow{AB}$.

5. Let k and m be parallel with a perpendicular distance of d between them and μ_k and μ_m be the mirror reflections over these lines. Prove that $\mu_k \circ \mu_m$ is a translation of length $2d$ in the direction perpendicular to k and m. [*Hint:* In Fig. 4.10 select the midpoint of A and $\mu_m(A)$, as well as another point on m. These points form congruent triangles with A and $\mu_m(A)$. Repeat with the line k. Analyze other cases similarly.] Also prove that $\mu_m \circ \mu_k$ and $\mu_k \circ \mu_m$ are inverses.

6. Let k and m intersect at point P and form an angle of $r°$ and μ_k and μ_m be the mirror reflections over these lines. Prove that $\mu_k \circ \mu_m$ is a rotation of $2r°$ around P. [*Hint:* In Fig. 4.11 let Q be the midpoint of A and $\mu_m(A)$. Use triangles $\triangle PAQ$ and $\triangle P\mu_m(A)Q$. Continue as in Problem 5. Decide what other cases, besides those in Fig. 4.11, can occur.] Also prove that $\mu_m \circ \mu_k$ and $\mu_k \circ \mu_m$ are inverses.

7. Let Q be between P and R on a Euclidean line. Explain why, for any isometry α, $\alpha(Q)$ is between $\alpha(P)$ and $\alpha(R)$ and all three are on a line.

8. Let ρ_1 and ρ_2 be any two rotations. Prove that their composition $\rho_1 \circ \rho_2$ is a translation, a rotation, or the identity. Find the conditions that are necessary and sufficient for the composition $\rho_1 \circ \rho_2$ to be a translation.

9. Let τ_1 and τ_2 be two translations and P and Q be two points. How are $\tau_2 \circ \tau_1$ and $\tau_1 \circ \tau_2$ related? Draw a figure showing P, Q, $\tau_1(P)$, $\tau_1(Q)$, $\tau_2(\tau_1(P))$, and $\tau_2(\tau_1(Q))$. Prove that the composition $\tau_1 \circ \tau_2$ is a translation. [*Hint:* Use SAS.]

10. Prove that **D**, the set of all direct isometries of the Euclidean plane, is a transformation group. Note that **D** preserves orientation in addition to all Euclidean properties.

11. Let **V** be the set of all Euclidean plane isometries that take vertical lines to vertical lines. Describe **V** and prove that it is a transformation group.

12. Define two sets $A = \{A_i : i \in I\}$ and $B = \{B_i : i \in I\}$ to be *congruent*, written $A \cong B$, iff for all $i, j \in I$, $d(A_i, A_j) = d(B_i, B_j)$.

a) Why are two triangles congruent under this definition also congruent under the usual definition? [*Hint:* Consider the vertices of the triangles.]

b) Why are any two lines congruent under this definition?

c) Why are circles with equal radii congruent under this definition?

13. Define two sets A and B to be *isometric* iff there is an isometry α such that $\alpha(A) = B$. The definition of congruent sets in Problem 12 guarantees that isometric sets are congruent. Show the converse in Euclidean geometry: For any two congruent Euclidean plane sets $A = \{A_i : i \in I\}$ and $B = \{B_i : i \in I\}$, there is an isometry taking A to B. [*Hint:* Use Theorem 4.2.3, its proof, and Theorem 4.2.5.]

4.3 ALGEBRAIC REPRESENTATION OF TRANSFORMATIONS

How can a computer display various viewpoints, zooming in and rotating as the user desires? Computer graphics software uses matrices and linear algebra extensively to compute such geometric transformations. (See Section 6.6 for more information, including the use of perspective.) This algebraic representation helped mathematicians, physicists, and others in many fields long before the advent of computers. Matrices and linear algebra also give a deeper insight into transformations. For convenience we often identify a transformation by its matrix.

If M is a matrix and P is a point, we write $Q = M \cdot P$ or $Q = MP$ to indicate that Q is the image of P by the transformation represented by M. This notation fits well with that of functions, $y = f(x)$, and follows standard linear algebra notation. However, it means that points are column vectors—for example $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, rather than the familiar ordered pair $(2, 3)$. Column vectors are awkward to print in the body of the text, so we will write the column vector $\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$ as $(x_1, x_2, \ldots, x_n)$. We use row vectors, such as $[x_1, x_2, \ldots, x_n]$, to represent lines.

Example 1 Recall that $\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ dx + ey \end{bmatrix}$. Thus the matrix $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ defines a transformation taking (x, y) to $(ax + by, dx + ey)$. The matrix $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents the rotation of 90° around the origin (Fig. 4.14). The matrix $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ doubles each point's distance from the origin (Fig. 4.15). The matrix $M = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$ represents the mirror reflection over the line $y = \frac{1}{2}x$ (Fig. 4.16). Select various points, such as $(-1, 1)$ and $(1, 2)$, and find their images under these matrices to verify the preceding statements. ●

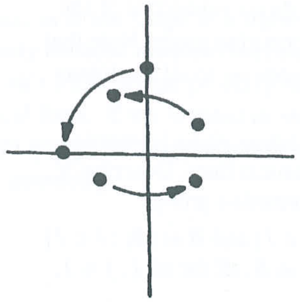

Figure 4.14

Exercise 1 In Example 1 find the product $R \cdot R$ and verify that it represents a rotation of 180° around the origin. Similarly, find $M \cdot M$ and verify that it represents the identity transformation. Find the inverse of D and verify that it halves each point's distance from the origin.

The preceding examples indicate that two-by-two matrices can represent a variety of plane transformations. However, they have a fatal drawback for our purposes: They all fix the origin (0, 0). Thus these matrices cannot represent nontrivial translations and many other transformations. For example, the translation $\tau(x, y) = (x + 3, y + 2)$ takes (0, 0) to (3, 2). The key difference of τ from the preceding examples is the addition of constants. Mathematicians have devised a simple way around this problem by using as their model the plane $z = 1$ in $\mathbf{R}^3$. Clearly, it has the same geometric properties as $\mathbf{R}^2$, which, in effect, is the plane $z = 0$. However, $z = 1$ has the key algebraic advantage

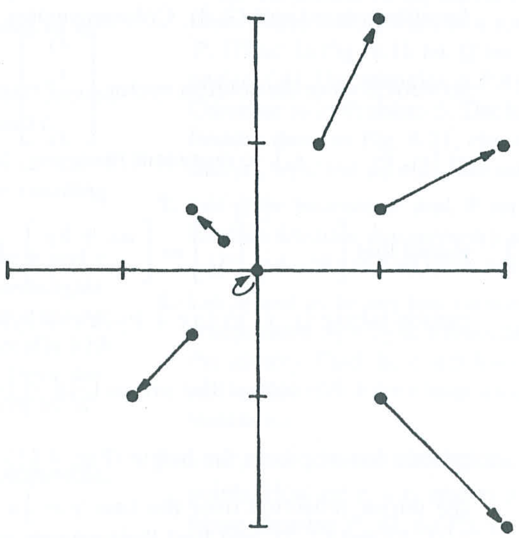

Figure 4.15

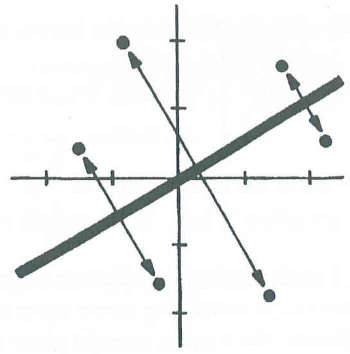

Figure 4.16

that all its points $(x, y, 1)$, including the new "origin" $(0, 0, 1)$, can be moved by 3×3 matrices. The third coordinate of $(x, y, 1)$ does not really "do" anything. For example, the distance between two points still depends only on their first two coordinates.

Interpretation By a *point* of the Euclidean plane we mean any triple $(x, y, 1)$, where x and y are real numbers. The *distance* between $(x, y, 1)$ and $(u, v, 1)$ is $\sqrt{(x-u)^2 + (y-v)^2}$.

Exercise 2 Verify that the matrix $\begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{bmatrix}$ maps $(x, y, 1)$ to $(ax + by, dx + ey, 1)$ and so corresponds to the same transformation given in Example 1 as $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$. Verify that the matrix $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ represents the translation $\tau(x, y, 1) = (x + 3, y + 2, 1)$.

It is no accident that the bottom row of both matrices in Exercise 2 is [0 0 1]. This restriction ensures that a general 3×3 matrix $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ maps the plane $z = 1$ to itself. That is, $M \cdot (x, y, 1)$ must equal $(\_, \_, 1)$. This forces M to have $g = h = 0$ and $i = 1$. A theorem of linear algebra states that a linear transformation is one-to-one and onto the whole space iff its matrix is invertible or, equivalently, the determinant is not zero. Hence an invertible 3×3 matrix represents a plane transformation provided that its bottom row is [0 0 1]. We call the transformations for the plane $z = 1$ *affine transformations* to distinguish them from linear transformations, which leave the origin of the space fixed.

Interpretation By *(plane) affine matrix* we mean any invertible 3×3 matrix whose bottom row is [0 0 1].

Exercise 3 Verify that an affine matrix leaves (0, 0, 1) fixed provided that the last column of the matrix is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Rewrite the matrices of Example 1 as affine matrices. Verify that $ae - bd$ is the determinant of $\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$.

Linear algebra provides an elegant interpretation of lines. Usually, a line is the set of points (x, y) satisfying some equation $ax + by + c = 0$. However, with triples $(x, y, 1)$ for points, the 1 has a natural place in that equation: $ax + by + c \cdot 1 = 0$. If we replace the left-hand side by the product of a row vector and a column vector, the equation $[a, b, c]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$ suggests that we can represent a line by three coordinates $[a, b, c]$. In $ax + by + c = 0$ if $b \neq 0$, we can solve the equation for y to get the familiar form (Exercise 4). If $b = 0$ and $a \neq 0$, we have a vertical line. (If $b = 0 = a$, the equation reduces to $c = 0$, which is not a line.)

Exercise 4 Verify that the familiar equation $y = mx + b$ corresponds to $[m, -1, b]$. Verify that vertical lines $x = c$ correspond to $[1, 0, -c]$.

Note that for any nonzero k, $[ka, kb, kc]$ represents the same line as $[a, b, c]$ because $[ka, kb, kc](x, y, 1) = k[a, b, c](x, y, 1)$. Technically, then, a line is "the equivalence class of all row vectors differing by a nonzero scalar." However, for convenience, we use a row vector $[a, b, c]$ as the name of a line. This interpretation of lines ensures that affine transformations map lines to lines and, in addition, that a general transformation of the plane taking lines to lines must be one of these affine transformations. We know that an affine matrix M moves a point P to MP, but where M takes a given line isn't obvious. Experiment with examples before reading the answer in Theorem 4.3.1.

Interpretation A *line* is a row matrix $[a, b, c]$ such that not both a and b are 0. The point $(x, y, 1)$ is *on* the line $[a, b, c]$ iff their product is 0: $ax + by + c \cdot 1 = 0$. Two row vectors represent the same line iff one is the product of the other by a nonzero scalar.

Exercise 5 Verify that the line $[-2, -1, 1]$ is on the points $(-1, 3, 1)$ and $(2, -3, 1)$. Note that the point $(-2, -1, 1)$ is on the lines $[-1, 3, 1]$ and $[2, -3, 1]$. There is a close relationship between points and lines in this model, which we explore more deeply in Chapter 6.

Theorem 4.3.1 The affine matrix M takes $[a, b, c]$ to the line $[a, b, c]M^{-1}$.

Proof. We need to show that, for any point $(x, y, 1)$ on $[a, b, c]$, the new point $M(x, y, 1)$ is on the proposed image of $[a, b, c]$, namely, $[a, b, c]M^{-1}$. The product of the new line and point is $[a, b, c]M^{-1} \cdot M(x, y, 1) = [a, b, c]I(x, y, 1) = [a, b, c](x, y, 1) = 0$. ■

4.3.1 Isometries

An affine matrix is readily determined by where it takes the three reference points $O = (0, 0, 1)$, $X = (1, 0, 1)$, and $Y = (0, 1, 1)$. Theorem 4.3.2 uses these points to describe which affine matrices are isometries.

Exercise 6 Verify that the images of the points O, X, and Y under the matrix $M = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$ are $O' = (c, f, 1)$, $X' = (a + c, d + f, 1)$, and $Y' = (b + c, e + f, 1)$.

Theorem 4.3.2 An affine matrix $M = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$ is an isometry iff $M = \begin{bmatrix} \cos\theta & -\sin\theta & c \\ \sin\theta & \cos\theta & f \\ 0 & 0 & 1 \end{bmatrix}$ or $M = \begin{bmatrix} \cos\theta & \sin\theta & c \\ \sin\theta & -\cos\theta & f \\ 0 & 0 & 1 \end{bmatrix}$, for some angle θ.

Proof. ($\Rightarrow$) If M is an isometry, then, for the three points O', X', and Y' of Exercise 6, $\triangle O'X'Y' \cong \triangle OXY$ (Fig. 4.17). The distance $d(O', X')$ is $\sqrt{a^2 + d^2}$. As M is an isometry, $a^2 + d^2 = 1$. Then for some angle θ, $a = \cos\theta$ and $d = \sin\theta$. Similarly, the distance $d(O', Y') = \sqrt{b^2 + e^2}$ forces $b = \cos\phi$ and $e = \sin\phi$, for some angle ϕ. Furthermore, $m\angle X'O'Y' = 90°$. As Fig. 4.17 illustrates, $\phi = \theta \pm 90°$. When $\phi = \theta + 90°$, the isometry is direct. When $\phi = \theta - 90°$, the isometry is indirect. Trigonometry gives $\sin(\theta \pm 90°) = \pm\cos\theta$ and $\cos(\theta \pm 90°) = \mp\sin\theta$.

($\Leftarrow$) See Problem 4. ■

Exercise 7 Verify that the determinant of $\begin{bmatrix} \cos\theta & -\sin\theta & c \\ \sin\theta & \cos\theta & f \\ 0 & 0 & 1 \end{bmatrix}$ is $+1$ and that the determinant of $\begin{bmatrix} \cos\theta & \sin\theta & c \\ \sin\theta & -\cos\theta & f \\ 0 & 0 & 1 \end{bmatrix}$ is -1.

Isometries split naturally into two classes, as given in Theorem 4.3.2. Exercise 7 shows that determinants identify these classes. The first class, with determinant $+1$, contains the direct isometries, for which θ is the angle of rotation. If $\theta = 0$, the direct

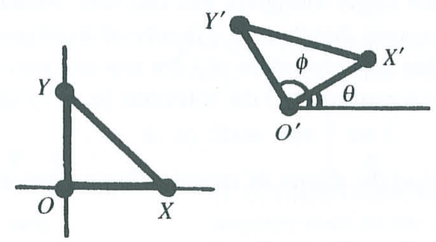

Figure 4.17

isometry is a translation. An isometry of the second class, with determinant -1, is an indirect isometry and its line of reflection makes an angle of $\theta/2$ with the x-axis.

Example 2 Find the center of rotation of $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and the line of reflection of $M = \begin{bmatrix} 0.6 & 0.8 & 2 \\ 0.8 & -0.6 & -4 \\ 0 & 0 & 1 \end{bmatrix}$. Which lines are stable under M?

Solution. The center of this rotation is a fixed point, say, $(u, v, 1)$, satisfying $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$. This equation reduces to $-v + 2 = u$ and $u + 3 = v$, or $u = -\frac{1}{2}$ and $v = 2\frac{1}{2}$. We can find the fixed points of M similarly, obtaining the pair of equations $-0.4u + 0.8v + 2 = 0$ and $0.8u - 1.6v - 4 = 0$. The second is a multiple of the first, so we get an infinite family of fixed points, $(u, \frac{1}{2}u - 2\frac{1}{2}, 1)$. That is, all the fixed points are on the line of reflection $[\frac{1}{2}, -1, -2\frac{1}{2}]$ or, more familiarly, $y = \frac{1}{2}x - 2\frac{1}{2}$. The other stable lines of M need a more general approach because for any nonzero multiple λ, $\lambda[a, b, c]$ is the same line as $[a, b, c]$. We need to solve $[a, b, c]M^{-1} = \lambda[a, b, c]$ but first have to find the possible values of λ. We need $[a, b, c]M^{-1} = \lambda[a, b, c] = \lambda[a, b, c]I$ or, equivalently, $[a, b, c](M^{-1} - \lambda I) = 0$. That is, the matrix $(M^{-1} - \lambda I)$ must have a determinant of zero. For $M = \begin{bmatrix} 0.6 & 0.8 & 2 \\ 0.8 & -0.6 & -4 \\ 0 & 0 & 1 \end{bmatrix} = M^{-1}$, $M^{-1} - \lambda I = \begin{bmatrix} 0.6-\lambda & 0.8 & 2 \\ 0.8 & -0.6-\lambda & -4 \\ 0 & 0 & 1-\lambda \end{bmatrix}$ and the determinant is $(1-\lambda)[(0.6-\lambda)(-0.6-\lambda) - (0.8)(0.8)] = (1-\lambda)(\lambda^2 - 1)$. Thus the possibilities are $\lambda = 1$ (as a double root) and $\lambda = -1$. The root $\lambda = -1$ gives the line of reflection, $[\frac{1}{2}, -1, -2\frac{1}{2}]$. For $\lambda = 1$, we get $[a, b, c]\begin{bmatrix} 0.6 & 0.8 & 2 \\ 0.8 & -0.6 & -4 \\ 0 & 0 & 1 \end{bmatrix} = [a, b, c]$, which reduces to $a = 2b$. This outcome gives a family of parallel lines $[2, 1, c]$ or, equivalently, $y = -2x - c$, all perpendicular to the line of reflection. Note that the double root gives a family of stable lines, whereas the single root gives just one line. Actually, the fact that $\lambda = 1$ is a double root for M ensures that there is a family of fixed points, as we found previously. Problem 7 shows that the only values of λ for any isometry are 1 and -1. In linear algebra the λ are called *eigenvalues*, and the solutions $[a, b, c]$ are called *eigenvectors*. ●

Example 3 Find the matrix M representing a rotation of θ around the point $(u, v, 1)$.

Solution. We build the rotation around $(u, v, 1)$ from the translation $T = \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}$

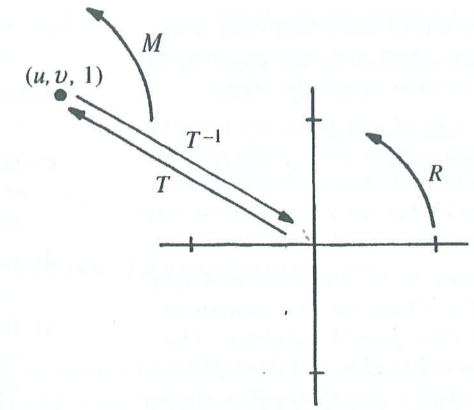

Figure 4.18

that moves $(0, 0, 1)$ to $(u, v, 1)$ and the rotation $R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ of θ around $(0, 0, 1)$. In effect we first move the point $(u, v, 1)$ to $(0, 0, 1)$, rotate there, and move back (Fig. 4.18). That is, we claim that $M = TRT^{-1}$. We verify that TRT^{-1} fixes $(u, v, 1)$ and is a direct isometry because its determinant is 1. Hence it is a rotation. Then we verify that the image of the point $(u + 1, v, 1)$ is $(u + \cos\theta, v + \sin\theta, 1)$. Explain why this solution shows the angle of rotation of TRT^{-1} to be θ. ●

PROBLEMS FOR SECTION 4.3

1. For each matrix, decide whether it is a translation, a rotation, a mirror reflection, or a glide reflection and find its fixed points and stable lines.

$$A = \begin{bmatrix} 0.8 & -0.6 & 2 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.8 & 0.6 & -1/3 \\ 0.6 & -0.8 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.8 & 0.6 & 2 \\ 0.6 & -0.8 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Repeat Problem 1 for the matrices $A \cdot B$, $B \cdot A$, $B \cdot C$, and $C \cdot D$.

3. a) Find the matrix for the rotation of 30° with a center of rotation of $(2, 3, 1)$.

b) Find the matrix of the mirror reflection over the line $y = 2x$.

c) Repeat part (b) for the line $y = 2x + 1$.

d) Find the matrices of all glide reflections over the line $y = 2x$.

4. Let $M_1 = \begin{bmatrix} \cos\theta & -\sin\theta & c \\ \sin\theta & \cos\theta & f \\ 0 & 0 & 1 \end{bmatrix}$,

$M_2 = \begin{bmatrix} \cos\theta & \sin\theta & c \\ \sin\theta & -\cos\theta & f \\ 0 & 0 & 1 \end{bmatrix}$, $P = (u, v, 1)$, and $Q = (s, t, 1)$. Show that $d(P, Q) = d(M_1P, M_1Q) = d(M_2P, M_2Q)$. [*Hint:* $\cos^2(\theta) + \sin^2(\theta) = 1$.]

5. a) Show that $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ switches $(1, 0, 1)$ and $(\cos\theta, \sin\theta, 1)$. Explain why this isometry must be the mirror reflection over the line through the origin that makes an angle of $\theta/2$ with the x-axis.

b) Let M and N be mirror reflections over lines through the origin. Show that their product in either order is a rotation around the origin.

c) In Theorem 4.3.2 let M and N be two general indirect isometries with the same angle θ. (Thus the last column of each matrix should have general variables.) What can you say about their product? Interpret this conclusion geometrically.

6. Write general matrices M for a rotation of θ and N for a rotation of ϕ. (Thus the last column of each matrix should have general variables.) Use trigonometry to show that MN and NM both represent rotations of $(\theta + \phi)$, unless $(\theta + \phi)$ is a multiple of 360°. What happens in this situation?

7. a) Prove that $\lambda = 1$ is always an eigenvalue of an affine matrix.

b) Prove that an indirect isometry has $\lambda = 1$ as a double eigenvalue and $\lambda = -1$ as an eigenvalue.

c) Prove that a matrix representing a direct isometry satisfies one of the following three situations, depending on the value of θ: (i) $\lambda = 1$ is a triple eigenvalue (θ is a multiple of 360°); (ii) $\lambda = 1$ is an eigenvalue, and $\lambda = -1$ is a double eigenvalue (θ is an odd multiple of 180°); or (iii) $\lambda = 1$ is an eigenvalue, and the other eigenvalues are complex (all other values of θ). Thus the only possible real eigenvalues of an isometry are $\lambda = 1$ and $\lambda = -1$.

8. The *central symmetry with respect to* A takes a point P to the point P', where A is the midpoint of $\overline{PP'}$ (Fig. 4.19).

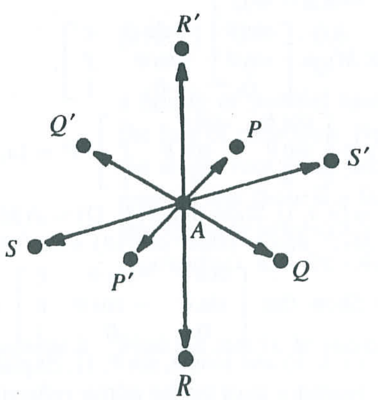

Figure 4.19

a) Find the matrix for the central symmetry with respect to $(0, 0, 1)$.

b) Repeat part (a) for $(u, v, 1)$. [*Hint:* Find the image of $(0, 0, 1)$.]

c) Verify that the matrix in part (b) has a determinant of $+1$ and its square is the identity. Explain what these algebraic properties mean geometrically.

d) Find the stable lines of the general central symmetry from part (b).

e) Find the composition of two central symmetries, one with respect to $(u, v, 1)$ and the other with respect to $(s, t, 1)$. Identify the type of isometry for this composition and explain what is happening geometrically. What happens when you switch the order?

f) Prove that a central symmetry sends every line to a line parallel to itself.

g) Prove that an isometry sends every line to a line parallel to itself iff that isometry is either a central symmetry or a translation.

h) Show that the set of translations and central symmetries form a transformation group. [Part (g) shows these isometries preserve the direction of a line.]

9. Let T be the matrix for a translation and M for an isometry. Explore the idea behind Example 3: TMT^{-1} represents a transformation essentially the same as M. (TMT^{-1} is called the *conjugate* of M by T.)

a) If M represents a rotation of $m°$ with any center, prove that TMT^{-1} is a rotation with the same angle.

b) If M represents the mirror reflection over the line k, prove that TMT^{-1} represents a mirror reflection by showing that its determinant is -1 and when multiplied by itself it gives the identity. Explain why these algebraic properties force TMT^{-1} to be a mirror reflection.

c) What can you say about TMT^{-1} if M is a translation?

10. a) Show the set of translations to be a transformation group.

b) Show the set of rotations fixing $(0, 0, 1)$ to be a transformation group.

c) Show the set of rotations fixing a point A to be a transformation group. [*Hint:* Use Example 3.]

d) Show the set of isometries fixing a given point A to be a transformation group.

e) Show the set of isometries leaving a line stable to be a transformation group.

f) Show that the indirect isometries are not a transformation group. Which properties for a transformation group fail?

11. a) Prove that an invertible matrix M never has $\lambda = 0$ as an eigenvalue.

b) If $\lambda \neq 0$ is an eigenvalue of an affine matrix M, prove that $1/\lambda$ is an eigenvalue of M^{-1}.