4.2 Isometries

The most important family of transformations, isometries, do not change the distance between points as the transformations move these points. Isometries are the dynamic counterpart to the Euclidean notion of congruence.

Definition 4.2.1 A transformation σ is an *isometry* on a set S with a distance function d iff for all points P and Q in S, $d(P, Q) = d(\sigma(P), \sigma(Q))$. If d is the usual distance on the Euclidean plane, then we call σ a *Euclidean plane isometry*.

Example 1 A rotation is an isometry (Fig. 4.4). It has one fixed point, in this case the origin. Most rotations have no stable lines. Describe the stable lines of a rotation of 180° around a point O. ●

Example 2 As shown in Fig. 4.5, σ doubles x-coordinates and halves y-coordinates. It is a transformation but not an isometry. ●

Exercise 1 Explain why an isometry takes a circle to a circle.

Example 3 A mirror reflection over a line k is an isometry (Fig. 4.6). Points on the line k are fixed. Any other point P is mapped to the point $\mu(P)$, where k is the perpendicular bisector of $\overline{P\mu(P)}$. The stable lines are the line of fixed points and the lines perpendicular to that line. ●

Exercise 2 In Example 3 explain why m is stable provided that $m \perp k$ or $m = k$.

Example 4 The isometry depicted in Fig. 4.7, called a translation, adds 3 to the x-coordinate and 2 to the y-coordinate of any point. A translation has no fixed points. The stable lines of a translation are parallel to the direction of the translation. ●

Definition 4.2.2 A Euclidean plane isometry τ is a *translation* iff, for all points P and Q, the points P, Q, $\tau(Q)$, and $\tau(P)$ form a parallelogram. (See Fig. 4.7.) The translation τ is said to be

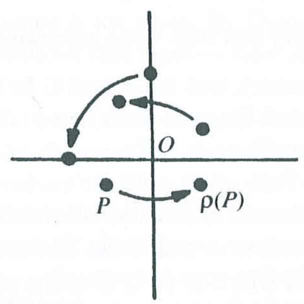


Figure 4.4

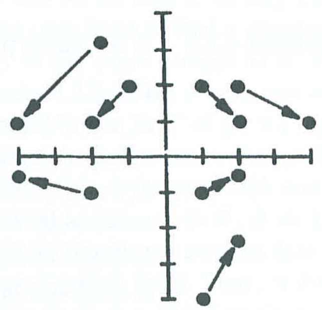


Figure 4.5

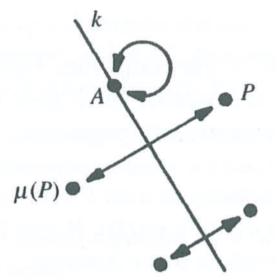


Figure 4.6

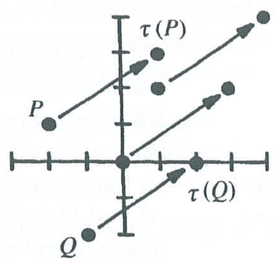


Figure 4.7

in the direction of $\overrightarrow{P\tau(P)}$. A Euclidean plane isometry ρ is a *rotation* of $r°$ iff there is a point O such that $\rho(O) = O$ and, for all other P, $m\angle PO\rho(P) = r°$. (See Fig. 4.4.) A Euclidean plane isometry μ is a *mirror reflection* over the line k iff, for every point P, k is the perpendicular bisector of the segment $\overline{P\mu(P)}$. (See Fig. 4.6.)

Exercise 3 Why is the identity both a translation and a rotation?

We now want to characterize all Euclidean plane isometries. Theorems 4.2.3, 4.2.5, and 4.2.7 give important geometric descriptions of isometries. Their proofs depend on the algebraic ideas of transformation groups, as well as on geometric properties. You might benefit from first exploring these ideas visually in Projects 1–4.

Theorem 4.2.1 The isometries of a set form a transformation group.

Proof. For closure, let α and β be isometries on a set S and show that $\alpha \circ \beta$ is an isometry. Let P and Q be any points in S. Then $d(P, Q) = d(\beta(P), \beta(Q)) = d(\alpha(\beta(P)), \alpha(\beta(Q)))$, showing $\alpha \circ \beta$ to be an isometry. Next, the identity, which fixes every point, preserves distance and so is an isometry. Finally, for an isometry α we show that its inverse α^{-1} is also an isometry. For P and Q in S, let $\alpha^{-1}(P) = U$ and $\alpha^{-1}(Q) = V$. We must show that $d(P, Q) = d(U, V)$. Because α is an isometry and the inverse of α^{-1}, $d(U, V) = d(\alpha(U), \alpha(V)) = d(P, Q)$. Thus the isometries form a transformation group. ■

Theorem 4.2.2 A Euclidean plane isometry that fixes three noncollinear points is the identity.

Proof. Let α be an isometry, and A, B, and C be three noncollinear points fixed by α, and D be any other point. We must show that $\alpha(D) = D$. Wherever $\alpha(D)$ is, it must satisfy three distance equations: $d(A, D) = d(A, \alpha(D))$, $d(B, D) = d(B, \alpha(D))$, and $d(C, D) = d(C, \alpha(D))$. Thus $\alpha(D)$ must be on three circles: one centered at A with radius AD, the second centered at B with radius BD, and the third centered at C with radius CD (Fig. 4.8). Because A and B are distinct, the first two circles intersect in at most two points, one of which is D. If D is the only intersection, we are done. But suppose that there is another point, say, E. Then $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{DE}$. However, C is not on $\overleftrightarrow{AB}$. Thus C cannot be the same distance from D and E. Hence $\alpha(D)$ cannot be E, forcing $\alpha(D) = D$. ■

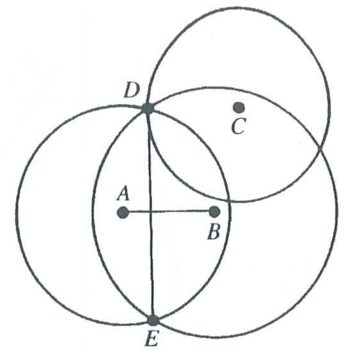


Figure 4.8

Theorem 4.2.3 A Euclidean plane isometry is determined by what it does to any three noncollinear points.

Proof. Let A, B, and C be any three noncollinear points, α be any isometry, and β be any isometry such that $\alpha(A) = \beta(A)$, $\alpha(B) = \beta(B)$ and $\alpha(C) = \beta(C)$. By Theorem 4.2.1, $\beta^{-1} \circ \alpha$ is an isometry and $\beta^{-1} \circ \alpha(A) = A$, $\beta^{-1} \circ \alpha(B) = B$ and $\beta^{-1} \circ \alpha(C) = C$. By Theorem 4.2.2, $\beta^{-1} \circ \alpha$ is the identity: $\beta^{-1} \circ \alpha = \iota$. When we compose both sides on the left with β, we get $\alpha = \beta$. ■

Theorem 4.2.4 For any two distinct points P and Q in the Euclidean plane, there is exactly one mirror reflection that takes P to Q.

Proof. With two distinct points we can use Euclid I-10 and I-11 to construct a perpendicular bisector. This perpendicular is unique by Hilbert's axiom III-4. Then the definition of a mirror reflection gives a unique mirror reflection, taking one point to the other. ■

Theorem 4.2.5 Every Euclidean plane isometry can be written as the composition of at most three mirror reflections.

Proof. Let α be any Euclidean plane isometry and A, B, and C be any three noncollinear points in the plane. By Theorem 4.2.3 we only have to find a composition of mirror reflections that together take A, B, and C to the same images as α does, say, P, Q, and R (Fig. 4.9). If $A \neq P$, then by Theorem 4.2.4, there is a mirror reflection μ_1 such that $\mu_1(A) = P$. Let $\mu_1(B) = B'$ and $\mu_1(C) = C'$. If $B' \neq Q$, we repeat this process, finding μ_2, which maps B' to Q. However, we need to prove that μ_2 leaves P fixed. Note that $d(P, Q) = d(A, B) = d(P, B')$. Thus P is on the perpendicular bisector of $\overline{QB'}$, which means that $\mu_2(P) = P$. Hence $\mu_2 \circ \mu_1$ moves A to P, B to Q, and C to some point C''. Finally, we need to move C'' to R. Again, we assume that $C'' \neq R$, use a mirror reflection μ_3, and verify that μ_3 leaves P and Q fixed. Thus, in the general case, $\mu_3 \circ \mu_2 \circ \mu_1$ maps A to P, B to Q, and C to R. If $A = P$, $B' = Q$, or $C'' = R$, then we omit μ_1, μ_2, or μ_3, respectively. The only case not covered by the argument presented is the identity transformation, ι. However, $\iota = \mu \circ \mu$, for any mirror reflection

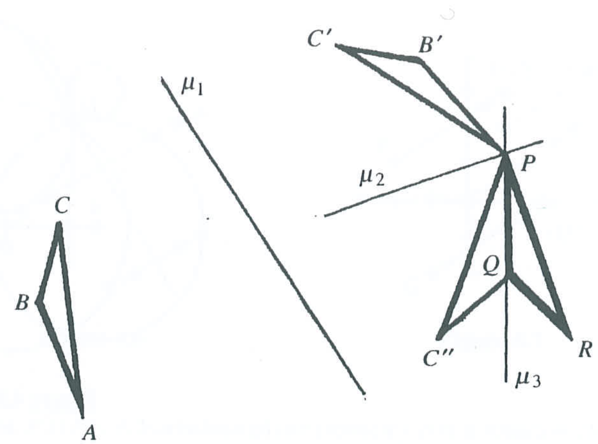


Figure 4.9

μ. Hence every Euclidean plane isometry can be written as the composition of three or fewer mirror reflections. ■

We can use Theorem 4.2.5 to classify the four possible types of Euclidean plane isometries, which are summarized in Theorem 4.2.7. We have already discussed three kinds of isometries: mirror reflections, rotations, and translations. Later we discuss the other kind of isometry, glide reflections. First, let's relate the three kinds of isometries we know to the composition of one, two, or three mirror reflections in Theorem 4.2.5. The "composition" of just one mirror reflection must be that mirror reflection.

For the composition of two mirror reflections there are three cases for the lines of reflection: The two lines are the same, they are distinct but parallel, and they intersect in a unique point. The first case gives the identity $\mu \circ \mu = \iota$. The second case gives a translation twice as long as the distance between the lines and in a direction perpendicular to them. Figure 4.10 illustrates several subcases. Problem 5 involves the use of congruent triangles to show the arrows in Fig. 4.10 are all parallel and the same length. The third case gives a rotation around the intersection of the two lines, where the angle is twice the angle between the lines. Figure 4.11 illustrates two subcases. Problem 6 involves the use of congruent triangles to show that the rotations in Fig. 4.11 are all the same angle and around the same point. A mirror reflection switches orientation. Because the identity, translations, and rotations are composed from two mirror reflections, they do not switch orientation and we say that they are *direct* isometries.

The remaining option, the composition of three mirror reflections, switches orientation three times and so is an *indirect* isometry, like a mirror reflection. Theorem 4.2.6 shows that three mirror reflections result either in a mirror reflection or a glide reflection, defined as follows.

Definition 4.2.3 A Euclidean plane isometry γ is a *glide reflection* iff there is a line k such that γ is the composition of the mirror reflection over k and a translation parallel to k.

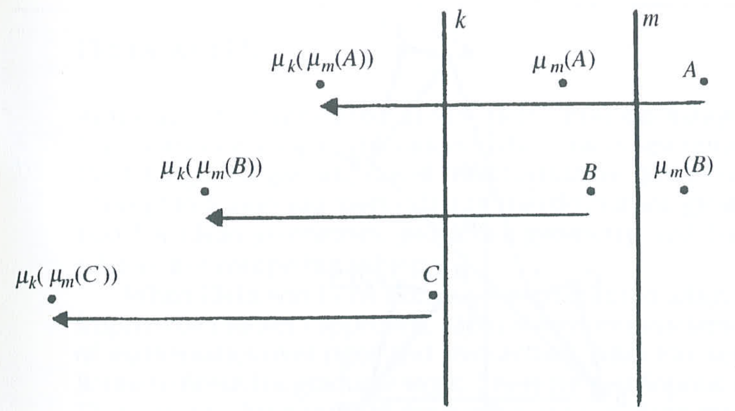


Figure 4.10

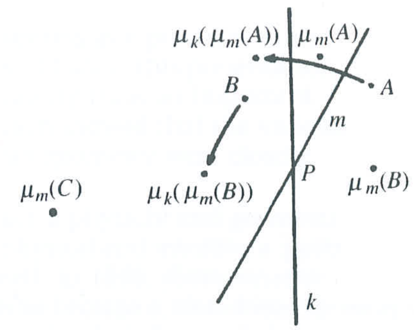


Figure 4.11

Example 5 Figure 4.12 indicates that a glide reflection can be written as the composition of a mirror reflection and a translation. ●

Exercise 4 Explain how a mirror reflection is a special case of a glide reflection. Explain why no other glide reflections have fixed points. Explain why only the line k is stable in other glide reflections.

Theorem 4.2.6 The composition of three mirror reflections is either a mirror reflection or a glide reflection.

Proof. By Theorem 4.2.3, the images of three noncollinear points determine an isometry. Let β be the composition of three mirror reflections and take A to A', B to B', and C to C'. Construct the midpoints M_1 and M_2 of the two line segments $\overline{AA'}$ and $\overline{BB'}$.

Case 1 Assume that $M_1 \neq M_2$. Let μ_k be the mirror reflection over the line $k = \overleftrightarrow{M_1M_2}$ (Fig. 4.13). Let X be the intersection of k with the line through A and $\mu_k(A)$. Then $\triangle AM_1X$ is similar to $\triangle AA'\mu_k(A)$ by Theorem 1.5.4. Theorem 1.5.1 then implies that $\overleftrightarrow{\mu_k(A)A'}$ is parallel to k. Similarly, $\triangle BM_2Y \sim \triangle BB'\mu_k(B)$ and $\overleftrightarrow{\mu_k(B)B'}$ is parallel to k, where Y is the intersection of k with $\overline{B\mu_k(B)}$. Thus $\overleftrightarrow{\mu_k(A)A'}$ is parallel to $\overleftrightarrow{\mu_k(B)B'}$. Note that $\triangle A'B'C' \cong \triangle\mu_k(A)\mu_k(B)\mu_k(C)$ because both are congruent with $\triangle ABC$.

Let τ be the translation taking $\mu_k(A)$ to A'.

Claim. τ takes $\mu_k(B)$ to B' and $\mu_k(C)$ to C', so $\beta = \tau\mu_k$, a glide reflection.

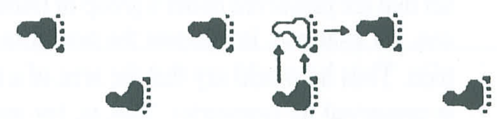


Figure 4.12

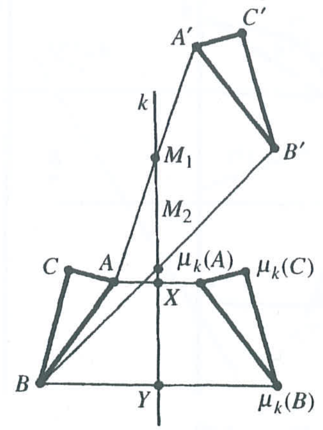


Figure 4.13

For the moment, let τ take $\mu_k(B)$ to B'' and $\mu_k(C)$ to C''. Then $\mu_k(B)$, B', and B'' are on the same line parallel to $\overleftrightarrow{\mu_k(A)A'}$. Only two points on $\overleftrightarrow{\mu_k(B)B'}$ have a distance from A' of $d(A', B'')$: B'' or the point Q, for which M_1 is the midpoint of $\overline{BQ}$. However, we assumed for case 1 that $M_1 \neq M_2$, so $B' \neq Q$. Because $d(A', B') = d(A', B'')$, $B' = B''$. By Problem 4 there are only two places for C'' to be, one of which is C'. However, the triangle $\Delta A'B'C''$ has the same orientation as $\Delta A'B'C'$, so $C'' = C'$. Thus β is the composition of μ_k with the translation τ. If $\mu_k(A) = A'$, then the translation is the identity and $\beta = \mu_k$. Otherwise β is a glide reflection.

Case 2 Assume that $M_1 = M_2$. If the midpoint of $\overline{CC'}$ is this same point, then the isometry is a rotation of 180° around the point, but that is a direct isometry and so is not β. Hence we can assume that the midpoint of A and A' differs from the midpoint of C and C'. Now we can apply the reasoning of case 1 here. ■

Theorem 4.2.7 There are four types of Euclidean plane isometries: mirror reflections, translations, rotations, and glide reflections.

Proof. From Theorem 4.2.5, every isometry can be written as the composition of one, two, or three mirror reflections. Verify that Problems 5 and 6, Theorem 4.2.6, and the preceding discussion cover all the possibilities for isometries. ■

4.2.1 Klein's definition of geometry

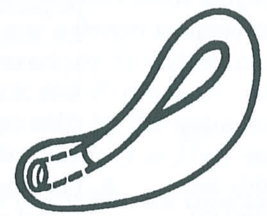
Felix Klein in his famous *Erlanger Programm*, given in 1872, used groups of transformations to give a definition of geometry: Geometry is the study of those properties of a set that are preserved under a group of transformations on that set. Klein realized that we can, for example, investigate the properties of Euclidean geometry by studying isometries. Thus he would say that the area of a triangle is a Euclidean property because area is preserved by isometries. That is, for any ΔA and any isometry σ, ΔA and $\sigma(\Delta A)$ have the same area. Under Klein's definition, congruence and measures of lengths and

FELIX KLEIN

At the age of 23 Felix Klein (1849–1925) gave his inaugural address as a professor at the University of Erlangen, the talk for which he is best remembered today. This presentation, the Erlanger Programm, raised transformation groups in geometry from an important concept to a unifying theme. Using transformation groups, Klein showed that the various non-Euclidean geometries, projective geometry, and Euclidean geometry were closely related, not competing subjects.

When Klein was 17 he became the assistant to Julius Plücker, a physicist and geometer. Inspired by Plücker's approach, Klein always emphasized the physical and intuitive a pects of mathematics over rigor and abstraction. After Plücker's death in 1868, Klein weıt to Berlin to finish his graduate work. There he met Sophus Lie, who became a close friend for many years. They went to Paris in 1870 for further studies. Both men were deeply influenced there by the possibility that group theory could unify mathematics. Indeed, Klein's Erlanger Programm in 1872 is a direct outgrowth of this inspiration.

Klein developed numerous theoretical models in geometry, including the Klein bottle, a curious two-dimensional surface with no inside and requiring four dimensions to realize it. The following figure illustrates that a three-dimensional representation of a Klein bottle intersects itself, unlike the theoretical shape.



A three-dimensional representation of a Klein bottle.

He distinguished single elliptic geometry from spherical geometry and investigated its models and transformations. To connect projective, Euclidean, and non-Euclidean geometries by using transformations, Klein developed the model of hyperbolic geometry named for him. He started to develop what we now call the Poincaré model, but he failed to see its connection to inversions that Henri Poincaré (1854–1912) found.

Klein was impressed with Poincaré's work and corresponded with him. However, their common interest soon became a fierce rivalry. Both produced important mathematics, but Klein suffered a nervous breakdown from the intense strain and felt that he had lost the contest. After recovering, Klein produced some mathematics and wrote several books. However, he focused on new tasks that called on his superior administrative abilities, building up mathematics research and education at his university, throughout Germany, and even in the United States.

angles are Euclidean properties, as is the shape of a figure. However, the orientation of figures isn't a Euclidean property because mirror reflections and glide reflections switch orientation. Also, verticality isn't a Euclidean property because some isometries, such as a rotation of 45°, tilt vertical lines. If we wanted to study orientation or verticality, we would need to use different groups of transformations, and, according to Klein, we would be studying a different geometry.

PROBLEMS FOR SECTION 4.2

1. Suppose that an isometry β takes $(1, 0)$ to $(-1, 0)$, $(2, 0)$ to $(-1, -1)$ and $(0, 2)$ to $(1, 1)$, respectively. Find the images of $(0, 0)$ and $(2, 2)$ and of a general point (x, y). Draw a figure showing these points and their images.

2. Outline the original placement of a small rectangular piece of paper on a larger piece of paper. Label the corners of both the small rectangle and the outline A, B, C, and D so that you can determine the rectangle's movements. Note that the centers of rotation are on the outline and do not move.

a) Rotate the small rectangle 180° around A and then 180° around C on the outline. Describe the resulting transformation.

b) Return the small piece of paper to its starting position and repeat part (a) but switch the order of the rotations. Describe how this new transformation differs from the one in part (a).

c) Repeat parts (a) and (b) but use rotations of 90° at A and C.

d) Repeat part (c) but rotate the rectangle 90° around A followed by a rotation of −90° around C.

e) Repeat part (c) with various angles and centers of rotations. Make a conjecture about the resulting transformations.

3. a) If μ_k is a mirror reflection over the line k and τ is a translation in the direction of k, investigate whether $\mu_k \circ \tau = \tau \circ \mu_k$ and justify your answer. [*Hint:* It may help to do this first physically with a triangle placed on a sheet of paper. Draw the line k on the paper. Geometer's Sketchpad or CABRI also will help.]

b) Find three mirror reflections whose composition is a glide reflection.

c) What is the composition of a glide reflection with itself? Justify your answer.

4. a) If α is an isometry which fixes two points, prove that α is the identity or the mirror reflection over the line through the fixed points.

b) If α and β are isometries such that $\alpha(A) = \beta(A)$ and $\alpha(B) = \beta(B)$, prove that $\alpha = \beta$ or $\alpha = \beta \circ \mu$, where μ is the mirror reflection over the line $\overleftrightarrow{AB}$.

5. Let k and m be parallel with a perpendicular distance of d between them and μ_k and μ_m be the mirror reflections over these lines. Prove that $\mu_k \circ \mu_m$ is a translation of length $2d$ in the direction perpendicular to k and m. [*Hint:* In Fig. 4.10 select the midpoint of A and $\mu_m(A)$, as well as another point on m. These points form congruent triangles with A and $\mu_m(A)$. Repeat with the line k. Analyze other cases similarly.] Also prove that $\mu_m \circ \mu_k$ and $\mu_k \circ \mu_m$ are inverses.

6. Let k and m intersect at point P and form an angle of $r°$ and μ_k and μ_m be the mirror reflections over these lines. Prove that $\mu_k \circ \mu_m$ is a rotation of $2r°$ around P. [*Hint:* In Fig. 4.11 let Q be the midpoint of A and $\mu_m(A)$. Use triangles $\triangle PAQ$ and $\triangle P\mu_m(A)Q$. Continue as in Problem 5. Decide what other cases, besides those in Fig. 4.11, can occur.] Also prove that $\mu_m \circ \mu_k$ and $\mu_k \circ \mu_m$ are inverses.

7. Let Q be between P and R on a Euclidean line. Explain why, for any isometry α, $\alpha(Q)$ is between $\alpha(P)$ and $\alpha(R)$ and all three are on a line.

8. Let ρ_1 and ρ_2 be any two rotations. Prove that their composition $\rho_1 \circ \rho_2$ is a translation, a rotation, or the identity. Find the conditions that are necessary and sufficient for the composition $\rho_1 \circ \rho_2$ to be a translation.

9. Let τ_1 and τ_2 be two translations and P and Q be two points. How are $\tau_2 \circ \tau_1$ and $\tau_1 \circ \tau_2$ related? Draw a figure showing P, Q, $\tau_1(P)$, $\tau_1(Q)$, $\tau_2(\tau_1(P))$, and $\tau_2(\tau_1(Q))$. Prove that the composition $\tau_1 \circ \tau_2$ is a translation. [*Hint:* Use SAS.]

10. Prove that **D**, the set of all direct isometries of the Euclidean plane, is a transformation group. Note that **D** preserves orientation in addition to all Euclidean properties.

11. Let **V** be the set of all Euclidean plane isometries that take vertical lines to vertical lines. Describe **V** and prove that it is a transformation group.

12. Define two sets $A = \{A_i : i \in I\}$ and $B = \{B_i : i \in I\}$ to be *congruent*, written $A \cong B$, iff for all $i, j \in I$, $d(A_i, A_j) = d(B_i, B_j)$.

a) Why are two triangles congruent under this definition also congruent under the usual definition? [*Hint:* Consider the vertices of the triangles.]

b) Why are any two lines congruent under this definition?

c) Why are circles with equal radii congruent under this definition?

13. Define two sets A and B to be *isometric* iff there is an isometry α such that $\alpha(A) = B$. The definition of congruent sets in Problem 12 guarantees that isometric sets are congruent. Show the converse in Euclidean geometry: For any two congruent Euclidean plane sets $A = \{A_i : i \in I\}$ and $B = \{B_i : i \in I\}$, there is an isometry taking A to B. [*Hint:* Use Theorem 4.2.3, its proof, and Theorem 4.2.5.]