4

Transformational Geometry

Computers store and reproduce intricate shapes such as the fractal pictured, relying on only a few numbers that somehow encode the shape. To do so they depend on geometric transformations to represent such shapes. In addition, computer-aided design (CAD) depends on transformations to present different views of an object. Transformational geometry underlies these applications and many other aspects of mathematics.

> Geometry is the study of those properties of a set which are preserved under a group of transformations on that set.
> —*Felix Klein*

> A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. —*G. H. Hardy*

4.1 Overview and History

Moving geometric figures around is an ancient and natural approach to geometry. However, the Greek emphasis on synthetic geometry and constructions and much later the development of analytic geometry overshadowed transformational thinking. The study of polynomials and their roots in the early nineteenth century led to algebraic transformations and abstract groups. At the same time, Augustus Möbius began studying geometric transformations. In the last third of the nineteenth century, Felix Klein and Sophus Lie showed the central importance of both groups and transformations for geometry. This approach enabled Klein and others to unify geometry at a time when new and different geometries seemed to be splitting this ancient discipline into competing theories. Transformations remained the dominant approach to geometry for 50 years. Transformations underlie the modern understanding of symmetry, which is essential in physics and chemistry, as well as mathematics. (See Chapter 5.) Early in the twentieth century physicists realized the power of transformations, starting with Einstein's theory of relativity and then with quantum mechanics. Although many geometric topics now transcend transformational geometry, this aspect of mathematics remains vital for understanding geometry.

We first investigate isometries, or the transformations that preserve distance. The proofs in Sections 4.1 and 4.2 are based on a synthetic approach, although we freely use coordinates in examples. Then the inclusion of linear algebra enables us to extend the types of transformations and to work in higher dimensions. Finally we discuss inversions, which are rich in geometric ideas and have important connections to complex analysis.

Example 1 The top and bottom of the shape depicted in Fig. 4.1 are mirror images. Matching points with their mirror images is one type of transformation, a mirror reflection. If the mirror is the x-axis, we can describe the transformation, say, μ, algebraically by $\mu(x, y) = (x, -y)$. The points on the x-axis remain fixed by μ: $\mu(x, 0) = (x, 0)$. Note that, if we perform the transformation twice, a point's image is mapped back to the original point. That is, $\mu(\mu(x, y)) = \mu(x, -y) = (x, y)$. We say that the entire shape is stable under μ because μ maps the shape to itself. ●

No amount of turning and twisting can turn a left hand into a right hand, even though they mirror each other. This condition is caused by the different *orientation* of an object and its mirror image. An (asymmetrical) three-dimensional object and its mirror image can't be superimposed on each other, even though the object and its

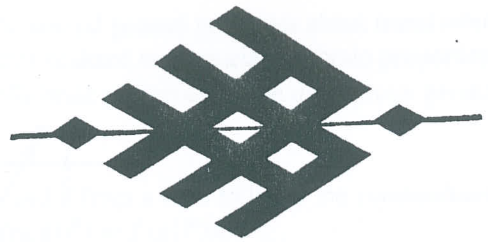

Figure 4.1

image are congruent. We particularly notice this change of orientation if we try to read the mirror image of a book. Rotations do not change orientation. Note that a two-dimensional mirror reflection, as described in Example 1, can be accomplished by a three-dimensional rotation of 180° of the entire space around the axis of fixed points. A two-dimensional mirror reflection switches orientation of points in the plane.

Definition 4.1.1 A *transformation* τ on a set S is a function from S to itself that is one-to-one and onto. That is, (i) [τ is a function] for every point P of S there is a unique point Q that is the image of P under $\tau : \tau(P) = Q$, and (ii) [τ is one-to-one and onto] for every point Q of S there is a unique point P for which Q is the image of P under τ. A point P is a *fixed point* of the transformation τ iff $\tau(P) = P$. A subset T of points in S is *stable* under the transformation τ iff the image of the subset T is again T, even if individual points of T move to other points in T. (Stable sets are often called *invariant* sets.)

The fixed points and stable sets of a transformation tell us important information about the transformation. For example, in Section 4.2 you will be able to recognize the type of an isometry by its fixed points and stable lines. Symmetry, the topic of Chapter 5, involves the study of transformations and their stable sets more deeply. In dynamical systems, a new field of mathematics, fixed points and stable sets help explain a much broader family of functions than we present here. (See Abraham and Shaw [1].)

Remark We don't need to separate the properties of one-to-one and onto here, although the difference is often important in mathematics. (See Gallian [5, 16] for a discussion of these concepts.)

Exercise 1 In Example 1 verify that all vertical lines are stable under μ even though individual points move. Verify that the x-axis also is stable under μ. Verify that any other horizontal line is not stable.

Example 2 On $\mathbf{R}^2$ define $\rho(x, y) = (y + 2, 2 - x)$ (Fig. 4.2). Show that ρ is a transformation. (Later we see ρ as a rotation.)

Solution. Each point (x, y) has a unique image $(y + 2, 2 - x)$, so ρ is a function. To show one-to-one and onto, we must start with any point (u, v) and show that there is a unique point (x, y) that ρ sends to (u, v). When we solve $u = y + 2$ and $v = 2 - x$, we find the solution: $(x, y) = (2 - v, u - 2)$. Because there is only one solution, ρ is by definition one-to-one and onto. ●

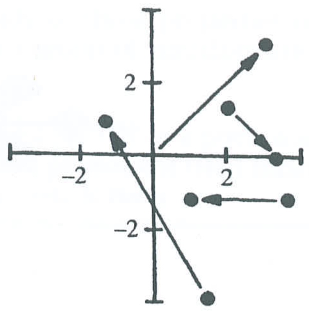

Figure 4.2

Exercise 2 Verify that ρ from Example 2 fixes the point $(2, 0)$. Why does ρ have no other fixed point? Assume that ρ is a rotation of 270° around $(2, 0)$. Why are circles with center $(2, 0)$ stable under ρ? Does ρ switch orientation?

Example 3 On $\mathbf{R}^2$ define $\psi(x, y) = (e^x, \sin y)$. Although ψ is a function, it is neither one-to-one nor onto. No matter what value of x we choose, e^x is positive, so ψ cannot be onto all of $\mathbf{R}^2$. Furthermore, the sine function is periodic, so two different points can map to the same point, demonstrating that ψ isn't one-to-one. ●

Example 4 The biologist D'Arcy Thompson used the idea of transformations in his study of comparative anatomy. Figure 4.3 reproduces one of his illustrations, depicting how he compared features of related species. The actual transformations he used go beyond the level of our study. (See Thompson [10].) ●

Example 5 On $\mathbf{R}^2$ let $\rho(x, y) = (y + 2, 2 - x)$ and $\mu(x, y) = (x, -y)$. The composition $\rho \circ \mu$ is given by $\rho \circ \mu(x, y) = \rho(\mu(x, y)) = \rho(x, -y) = (-y + 2, 2 - x) = (2 - y, 2 - x)$. We can show $\rho \circ \mu$ to be a transformation, as in Example 2. ●

Composing functions enables us to build and study a wide variety of functions in calculus, geometry, and other areas of mathematics. Theorem 4.1.1 shows that the composition of two transformations on a set is again a transformation on the set. This

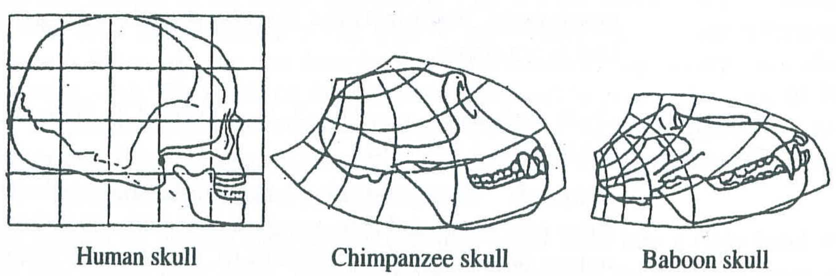

Figure 4.3 Comparing skulls by using transformations that distort coordinates.

property is the first of several general properties about transformations that nineteenth century mathematicians realized were crucial algebraic properties. In later sections we describe how these algebraic properties enable us to prove geometric properties about transformations.

Definition 4.1.2 For two functions f and g from a set S to itself, the *composition* of g followed by f, $f \circ g$ is defined by $f \circ g(P) = f(g(P))$.

Theorem 4.1.1 If α and β are any two transformations on S then $\alpha \circ \beta$ is a transformation on S.

Proof. Let α and β be two transformations on S and $P \in S$. For $\alpha \circ \beta$ to be a function, there must be a unique $R \in S$ such that $\alpha \circ \beta(P) = R$. Now β is a transformation, so there is a unique $Q \in S$ such that $\beta(P) = Q$. Similarly, there is a unique $R \in S$ such that $\alpha(Q) = R$. Hence R is the unique image of P under $\alpha \circ \beta$.

Let $V \in S$. For $\alpha \circ \beta$ to be one-to-one and onto, there must be a unique $T \in S$ such that $\alpha \circ \beta(T) = V$. Because α is a transformation, there is a unique $U \in S$ such that $\alpha(U) = V$. Similarly, there is a unique $T \in S$ such that $\beta(T) = U$. This result implies that T is the only element of S which $\alpha \circ \beta$ takes to V. Thus $\alpha \circ \beta$ is a transformation. ■

Example 6 For any set S, define the *identity* transformation by $\iota(P) = P$. For any transformation α on S, $\alpha \circ \iota = \alpha = \iota \circ \alpha$. The identity may seem of little importance by itself, but its presence simplifies investigations about transformations, just as the number 0 simplifies addition of numbers. ●

Example 7 On $\mathbf{R}^2$, if we compose $\rho(x, y) = (y + 2, 2 - x)$ and $\psi(x, y) = (2 - y, x - 2)$, then $\rho \circ \psi(x, y) = \rho(2 - y, x - 2) = ((x - 2) + 2, 2 - (2 - y)) = (x, y)$. Thus $\rho \circ \psi$ equals the identity, ι: ρ undoes what ψ did. ●

Definition 4.1.3 A transformation β is the *inverse* of a transformation α iff $\alpha \circ \beta = \iota$ and $\beta \circ \alpha = \iota$. We write α^{-1} for the inverse of α.

Exercise 3 In Example 7, verify that $\psi \circ \rho$ is also the identity. In Example 5, verify that $\rho \circ \mu$ is its own inverse.

Theorem 4.1.2 Every transformation α on a set S has a unique inverse, α^{-1}, which is a transformation satisfying $\alpha^{-1}(Q) = P$ iff $\alpha(P) = Q$.

Proof. Note that parts (i) and (ii) of the definition of a transformation (Definition 4.1.1) are closely related. This relationship implies that α^{-1}, as given in Theorem 4.1.2, is a function because α is one-to-one and onto. Similarly, α^{-1} is one-to-one and onto because α is a function. Hence α^{-1} is a transformation. Verify that $\alpha \circ \alpha^{-1} = \iota = \alpha^{-1} \circ \alpha$. To show uniqueness, we suppose that β also is an inverse of α and show that $\beta = \alpha^{-1}$. Let Q be any element of S. Because α is a transformation, there is a unique P such that $\alpha(P) = Q$. Then $\beta(Q) = P = \alpha^{-1}(Q)$. As α^{-1} and β agree everywhere, they are equal. ■

Various sets of transformations correspond to important geometric properties and also form groups, which are structures of great importance in mathematics. We define transformation groups because these are the only groups that we consider.

Remark Associativity plays a key role in group theory, but it always holds for composition of functions. Hence we can omit it from the definition. (See Gallian [5,17].)

Definition 4.1.4. A set **T** of transformations on a set S is a *transformation group* iff the following properties obtain.

i) (*closure*) The composition of two transformations in **T** is in **T**.
ii) (*identity*) The identity transformation is in **T**.
iii) (*inverses*) If a transformation τ is in **T**, then the inverse τ^{-1} is in **T**.

Theorem 4.1.3 The set of all transformations on a set is a transformation group.

Proof. See Theorems 4.1.1 and 4.1.2 and Example 6. ∎

Exercise 4 Verify that $\mu \circ \rho$ does not equal $\rho \circ \mu$ in Example 5.

PROBLEMS FOR SECTION 4.1

1. On **R**, the real numbers, define $\alpha(x) = x^3$ and $\beta(x) = 2x - 1$.

a) Show that α and β are transformations.

b) Find $\alpha \circ \beta$ and $\beta \circ \alpha$. Show that $\alpha \circ \beta \neq \beta \circ \alpha$.

c) Find α^{-1} and β^{-1}. Graph α and α^{-1} together. Repeat for β and β^{-1}. Describe the relationship between the graph of a transformation on **R** and the graph of its inverse.

d) Find $\alpha^{-1} \circ \beta^{-1}$ and $\beta^{-1} \circ \alpha^{-1}$. Which is the inverse of $\alpha \circ \beta$? Verify your answer.

2. For each of the following functions on $\mathbf{R}^2$, show on a graph what it does to various points, show that it is a transformation, and find its fixed point(s) and stable line(s).

a) $\omega(x, y) = (2 - x, 4 - y)$.

b) $\mu(x, y) = (y - 1, x + 1)$.

c) $\sigma(x, y) = (2x, 2y)$.

d) $\psi(x, y) = (\frac{1}{2}x + 1, \frac{1}{2}y - 1)$.

3. If α and β are transformations on a set S, prove that both $\alpha^{-1} \circ \beta^{-1}$ and $\beta^{-1} \circ \alpha^{-1}$ are transformations. Which of these two transformations is the inverse of $\alpha \circ \beta$? Prove your answer.

4. a) On $\mathbf{R}^2$ define σ by $\sigma(x, y) = (\frac{x}{2} + 2, \frac{y}{2} - 1)$. Find the fixed point of σ and call it F.

b) Let P_0 be any point in $\mathbf{R}^2$ and define the sequence $\{P_0, P_1, P_2, \ldots\}$ by $P_{n+1} = \sigma(P_n)$. Graph the sequence $\{P_0, P_1, P_2, \ldots\}$ for several initial choices of P_0. What happens in each case?

c) Repeat parts (a) and (b) for $\phi(x, y) = (2x + 2, 2y - 4)$.

d) Repeat parts (a) and (b) for $\rho(x, y) = (3 - y, x - 1)$.

The study of dynamical systems involves finding the long-term result of repeated application of a function. The fixed point of σ is called an *attracting* (or *stable*) *fixed point* because σ takes all points closer to the fixed point. The fixed point of ϕ is called a *repelling* (or *unstable*) *fixed point* because ϕ sends other points farther from the fixed point. The transformation ρ is said to be *periodic of period 4*, as four applications of ρ give the identity. These terms describe the dynamics of many familiar transformations. (See Abraham and Shaw [1].)

e) The transformation $\gamma(x, y) = (x + 2, -y)$ doesn't fit any of the preceding dynamics. Describe its long-term dynamics.