a) $d_H(A, B) = d_H(B, A) \geq 0$ and $d_H(A, B) = 0$ iff $A = B$.

b) If B is between A and C on the diameter AC, then $d_H(A, B) + d_H(B, C) = d_H(A, C)$.

10. a) Find the general pattern for the x-coordinates of the points P_i in Example 2.

b) Use part (a) to show that $d_H(P_i, P_{i+1}) = d_H(P_0, P_1)$.

3.5 Spherical and Single Elliptic Geometries

In one sense, mathematicians have studied the geometry of the sphere for millennia. However, before Bernhard Riemann in 1854 no one had thought of spherical geometry as a separate geometry, but only as properties of a Euclidean figure. The characteristic axiom of spherical geometry is that every two lines (great circles) always intersect in two points. (See Section 1.6.)

To retain the familiar notion of Euclidean and hyperbolic geometries that two points determine a line, Felix Klein in 1874 saw the need to modify spherical geometry. The usual way to do so was to identify opposite points on the sphere as the same point and study this "collapsed" geometry, which Klein called *single elliptic geometry*. Thus the characteristic axiom of single elliptic geometry is that every two distinct lines intersect in only one point. (Klein called spherical geometry *double elliptic geometry* because lines intersect in two points.) Spherical and single elliptic geometries share many theorems in common, such as the angle sum of a triangle is greater than 180°. In addition, single elliptic geometry possesses some unusual features worth noting. We can represent single elliptic geometry as the half of a sphere facing us (Fig. 3.37) so long as we remember that a line (or curve) that leaves the part facing us immediately reappears directly opposite because opposite points are identified.

A line in either of these geometries has many of the same properties as a circle in Euclidean geometry. First, we can't determine which points are "between" two points because there are two ways to go along a line from one point to another point. Note that we can use two points to "separate" two other points (Fig. 3.38). Second, the total length of a line is finite. A single elliptic line has another, more unusual property: It doesn't separate the whole geometry into two parts, unlike lines in Euclidean, hyperbolic, and spherical geometries. Figure 3.37 indicates how to draw a path connecting any two points not on a given line so that the path does not cross that line.

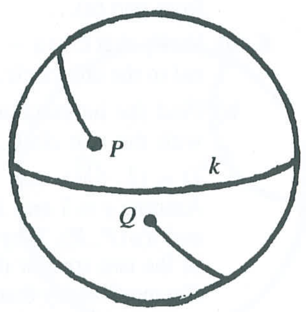

Figure 3.37 In the single elliptic geometry there is a path from P to Q that does not intersect k.

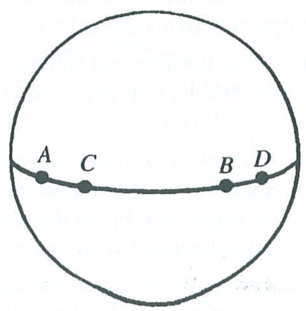

Figure 3.38 A and B separate C and D.

In certain ways, Euclidean geometry is intermediate between spherical and single elliptic geometries on the one hand and hyperbolic geometry on the other hand. For example, in Euclidean geometry, the angle sum of a triangle always adds to 180°. As we know in hyperbolic geometry, the corresponding sum falls short of 180° in proportion to the area of the triangle. In spherical and single elliptic geometries, this sum is always more than 180° and the excess is proportional to the area of a triangle. (Theorem 1.6.3 shows this condition for Euclidean spheres.) Indeed, in these geometries triangles can have three obtuse angles, so the sum can approach 540°.

In our development of hyperbolic geometry we assumed that Euclid's first 28 propositions hold, for they used only Euclid's first four postulates, but not the fifth postulate. Many of these propositions, including two of the triangle congruence theorems (SAS and SSS), continue to hold in spherical and single elliptic geometries.* However, most of the propositions after I-15, including AAS, do not hold in these geometries, even though they do not depend on the fifth postulate.

Figure 3.39 illustrates Euclid's approach to showing, as I-16 states, that in any triangle an exterior angle, such as $\angle BCD$, is larger than either of the other two interior angles, $\angle ABC$ and $\angle BAC$. From the midpoint E of $\overline{BC}$, Euclid extended AE to F so that $\overline{EF} \cong \overline{EA}$. Then by SAS $\triangle ECF \cong \triangle EBA$. He then concluded that $\angle BCD$ is larger than $\angle ECF$, which is congruent to the interior angle $\angle EBA$. Figure 3.39 supports this conclusion, but the similar situation shown in Fig. 3.40 for single elliptic geometry reveals that the conclusion depends on implicit assumptions. In Fig. 3.40, the part of $\overleftrightarrow{AE}$ that looks like segment $\overline{AE}$ covers more than half the length of the line. Hence the corresponding part of $\overleftrightarrow{EF}$ overlaps this apparent segment. Euclid implicitly assumed that lines extend infinitely in each direction. Postulate 2 only says, "to produce a finite straight line continuously in a straight line." The overlapping "segments" $\overline{AE}$ and $\overline{EF}$ in Fig. 3.40 satisfy the letter and, within reason, the spirit of postulate 2. Nevertheless, I-16 is false here because $\angle BCD$ can be smaller than $\angle ECF$.

Exercise 1 Draw the figure in spherical geometry corresponding to the situation depicted in Fig. 3.40.

* For SAS to hold we need to assume that a side is the shortest part of a geodesic.

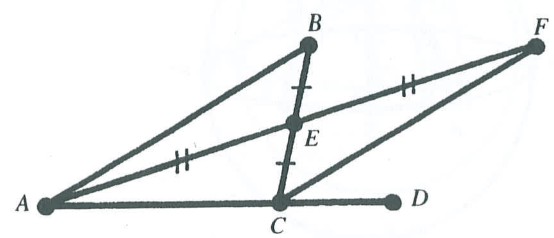

Figure 3.39 Euclid's diagram for proposition I-16.

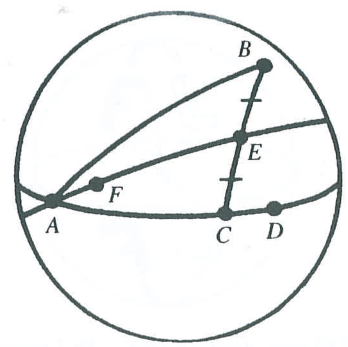

Figure 3.40 The diagram from Fig. 3.39 in single elliptic geometry.

We partially develop single elliptic geometry as we did hyperbolic geometry. (For a more thorough development, see Gans [5].) Theorems 3.2.1–3.2.8 do not relate to geometries where all lines intersect. Therefore we start with Theorem 3.3.1 concerning Saccheri quadrilaterals, which we repeat here.

Theorem 3.3.1 The summit angles of a Saccheri quadrilateral are congruent. The base and summit are perpendicular to the line on their midpoints.

Exercise 2 Verify that the proof of Theorem 3.3.1 holds in spherical and single elliptic geometry.

In hyperbolic geometry, Theorem 3.3.2 showed that the summit angles of a Saccheri quadrilateral were acute. Theorem 3.5.2 shows they are now obtuse. Theorem 3.5.1 provides a key step to this end.

Exercise 3 In Euclidean geometry what can you say about the summit angles of a Saccheri quadrilateral?

Theorem 3.5.1 In single elliptic geometry, all lines perpendicular to a given line intersect in one point.

Proof. From Theorem 3.3.1 we know that, in Saccheri quadrilateral $ABCD$, $\overline{EF}$ is perpendicular to both $\overline{AB}$ and $\overline{CD}$. By the characteristic property of single elliptic geometry, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in a unique point, say, P.

Claim. $d(A, P) = d(B, P)$ and $d(C, P) = d(D, P)$. (In a model, the distance is along the shortest path. As shown in Fig. 3.41, the shortest path from B to P goes "around behind" because we can "jump" from the right edge to the left edge.)

For a contradiction, construct P' on $\overleftrightarrow{AB}$ so that $\overline{AP} \cong \overline{BP'}$ and P and B separate A and P'. (Intuitively P' is on the "other" side of B, as depicted in Fig. 3.41.) Then $\triangle ADP \cong \triangle BCP'$ by SAS. Thus $\angle BCP'$, which is congruent to $\angle ADP$, is supplementary to $\angle BCD$. Then by Euclid I-14, $\overleftrightarrow{CP'}$ is the same line as $\overleftrightarrow{CD}$. This result would give two points of intersection of $\overleftrightarrow{AB}$ with $\overleftrightarrow{CD}$, which is a contradiction. So $P' = P$,

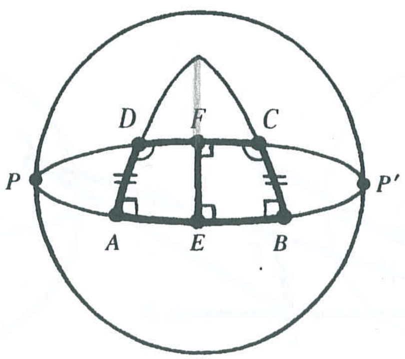

Figure 3.41

showing that $d(A, P) = d(B, P)$ and $d(C, P) = d(D, P)$. Furthermore, the distance from E to P is the same by way of A as by way of B, which is the maximum distance two points can be separated. The same holds for the distance from F to P, and $d(E, P) = d(F, P)$ by Euclid I-6, for isosceles triangles. Moreover, note that $d(E, P)$ and $d(F, P)$ do not depend on the length of $\overline{EF}$. That is, common perpendiculars are always the same length and intersect in the same point. ■

Theorem 3.5.2 In single elliptic geometry, the summit angles of a Saccheri quadrilateral are obtuse.

Proof. In Saccheri quadrilateral $ABCD$ construct $\overleftrightarrow{DG}$ perpendicular to $\overline{AD}$ and let Q be the intersection of $\overleftrightarrow{DG}$ and $\overleftrightarrow{AB}$ (Fig. 3.42). Then distance $d(A, P)$ is less than or equal to $d(A, Q)$ because Q is the point furthest from A by Theorem 3.5.1. If these distances were equal, $P = Q$, implying that $A = E$ and giving a trivial Saccheri quadrilateral. Hence $d(A, P) < d(A, Q)$. Thus $\overleftrightarrow{DP}$ enters $\triangle QDA$ at D, and so $\angle PDA$ must be smaller than $\angle QDA$, which is a right angle. Hence $\angle ADC$, a summit angle and supplementary to $\angle PDA$, is obtuse. ■

Once we have Theorem 3.5.2, the development of single elliptic geometry follows the lines of hyperbolic geometry. (See Project 5.)

Single elliptic geometry has another property worth discussing. Consider what happens to an asymmetrical object in single elliptic geometry as it moves along the path of a line (Fig. 3.43). This object actually has two opposite representations, as the "front" one slides out of view, it is "replaced" by the "rear" one, which is fundamentally different. The rear representation won't match the front representation because it has an opposite orientation. That is, these two representations are mirror images. No matter how you twist and turn two objects with opposite orientations in Euclidean geometry, as long as you stay in the plane, you cannot make them coincide. In single elliptic geometry these representations are the same, so we have the curious property that simply moving an object around in this geometry can switch its orientation. Single elliptic geometry is topologically a *nonorientable* surface, whereas spherical, Euclidean, and hyperbolic geometries are *orientable*. On a nonorientable surface there is no consistent way to define a clockwise direction.

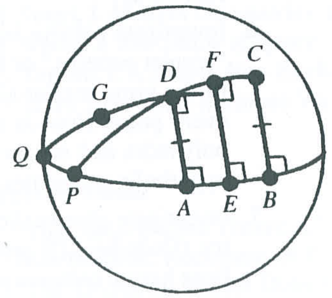

Figure 3.42

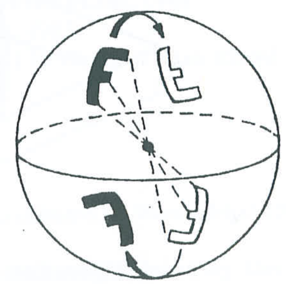

Figure 3.43 Single elliptic geometry is not orientable.

PROBLEMS FOR SECTION 3.5

1. Call a triangle with two right angles a *doubly right triangle*.
   a) Prove in single elliptic geometry that two doubly right triangles are congruent if their sides between the right angles are congruent.
   b) Prove in single elliptic geometry that two doubly right triangles are congruent if their third angles are congruent.
2. Euclid I-26 has two congruence theorems: AAS and ASA.
   a) Explain why Theorem 3.5.1 shows that AAS is not a triangle congruence theorem in single elliptic and spherical geometries.
   b) Prove that ASA is a triangle congruence theorem in single elliptic and spherical geometries, using SAS and a proof by contradiction.
3. Reformulate and prove Theorem 3.5.1 for spherical geometry.
4. Prove Theorem 3.5.2 in spherical geometry.
5. Do Problem 1 in spherical geometry.
6. Assume the analog to Theorem 3.4.5 in single elliptic and spherical geometries: The area of a triangle is proportional to the *excess* of the angle sum of a triangle over 180°. Each of the three angles of a triangle must have a measure of less than 180°.
   a) What would be the "triangle" if all three angles had a measure of 180°? What would be the area of this "triangle" if the sphere has radius r?
   b) Explain why the proof of Theorem 1.6.3 isn't sufficient to establish the analog to Theorem 3.4.5.

PROJECTS FOR CHAPTER 3

1. Construct an empirical model of hyperbolic geometry, using two regular mirrors and one convex mirror, as illustrated in Fig. 3.44. To make this curved mirror, lay some flexible reflective surface, such as mylar, on the outside of a cylindrical frame. Place some design inside the mirrors. You should see replicas of your design as though it were a part of hyperbolic geometry. As the angles at A, B, and C change, you will get different numbers of copies of your design around these points.

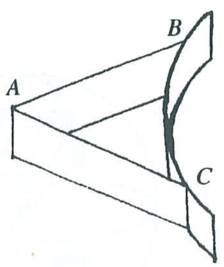

Figure 3.44

2. Use appropriate software (for example, Geometer's Sketchpad or CABRI) to create the Poincaré model of hyperbolic geometry. Investigate how the angle sum of a hyperbolic triangle changes as the size of the triangle changes. Construct a quadrilateral that has four interior angles of 60°.
3. In Euclidean geometry develop and prove the analogs, if any, to Theorems 3.3.3–3.4.6 using any of Euclid's propositions in Appendix A. How do they differ from the non-Euclidean cases?
4. In spherical geometry develop and prove the analogs to Theorems 3.3.3–3.4.6 using Problem 4 of Section 3.5.
5. In single elliptic geometry develop and prove the analogs to Theorems 3.3.3–3.4.6 using Theorem 3.5.2.
6. Investigate patterns in spherical geometry. Find all "regular patterns," or those where the same number of the same regular spherical polygons fit around every point. How do these patterns compare with polyhedra and regular patterns in Euclidean and hyperbolic geometries?
7. Investigate ultraparallel lines in hyperbolic geometry. (Cederberg [2] proves that any two ultraparallel lines have a common perpendicular.)
8. Investigate *limiting curves* or *horocycles*, the points at a constant distance from a hyperbolic line. These sets act like circles of infinite radius. (See Smart [12].)

9. Investigate the historical development of non-Euclidean geometry. (See Bonola [1], Kline [6, Chapter 36] and Richards [9].)
10. Investigate other non-Euclidean geometries. (See Yaglom [14].)
11. Investigate the concept of space. (See, for example, Rucker [10] and Weeks [13].)
12. Investigate curvature and differential geometry. (See McCleary [7].)
13. Write an essay comparing Euclidean and hyperbolic geometry. Address your essay to someone at the level of calculus.
14. Write an essay comparing how models and synthetic development have helped you understand hyperbolic geometry. Discuss how convincing the proofs in hyperbolic geometry are to you.
15. Write an essay discussing the difference between spherical geometry as a separate geometry and the geometry of the sphere as a part of Euclidean geometry.
16. Write an essay discussing whether you think that mathematics was discovered or invented. Use your understanding of non-Euclidean geometries as an example. (See, for example, Kline [6, 1032–1038].)

Suggested Readings

[1] Bonola, R. *Non-Euclidean Geometry: A Critical and Historical Study of its Development*. Mineola, N.Y.: Dover, 1955.
[2] Cederberg, J. *A Course in Modern Geometries*. New York: Springer-Verlag, 1989.
[3] do Carmo, M. *Differential Geometry of Curves and Surfaces*. Englewood Cliffs, N.J.: Prentice-Hall, 1976.
[4] Dunham, D. Hyperbolic symmetry. *Comp. and Maths. with Appl.*, 1986, 12(1 and 2):139–153.
[5] Gans, D. *An Introduction to Non-Euclidean Geometry*. New York: Academic Press, 1973.
[6] Kline, M. *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press, 1972.
[7] McCleary, J. *Geometry from a Differentiable Viewpoint*. Port Chester, N.Y.: Cambridge University Press, 1994.
[8] Moise, E. *Elementary Geometry from an Advanced Standpoint*. Reading, Mass.: Addison-Wesley, 1974.
[9] Richards, J. *Mathematical Visions: The Pursuit of Geometry in Victorian England*. San Diego: Academic Press, 1988.
[10] Rucker, R. *Geometry, Relativity and the Fourth Dimension*. Mineola, N.Y.: Dover, 1977.
[11] Sacks, O. *An Anthropologist on Mars*. New York: Knopf, 1995.
[12] Smart, J. *Modern Geometries*. Monterey, Calif.: Brooks/Cole, 1988.
[13] Weeks, J. *The Shape of Space*. New York: Dekker, 1985.
[14] Yaglom, I. *A Simple Non-Euclidean Geometry and Its Physical Basis* (transl. A. Shenitzer). New York: Springer-Verlag, 1979.

Suggested Media

1. "The Gauss-Bonnet Theorem," 25-minute film, distributed by Ward's Modern Learning Aids Division, Rochester, N.Y., 1974.
2. "The Geometry Euclid Didn't Know," 16-minute film, distributed by David Nulsen Enterprises, Santa Monica, Calif., 1979.
3. "Hungarian Rhapsody: Janos Bolyai and his Revolutionary Discovery of Non-Euclidean Geometry," Vol. 3 of *Rebels Who Count*, 30-minute video, Media Center, University of Toronto, 1991.

4. "The Majestic Clockwork," 52-minute video, from *The Ascent of Man* series, distributed by Time–Life Multimedia, New York, 1974.
5. "Non-Euclidean Geometry," 24-minute video, Films for the Humanities and Sciences, Princeton, N.J., 1996.
6. "A Non-Euclidean universe," 25-minute film, distributed by University Media, Solana Beach, Calif., 1978.
7. "Not Knot," 20-minute video and supplement, University of Minnesota Geometry Center, distributed by Jones and Bartlett , Boston, Mass., 1991.
8. "Prince of Mathematicians: Karl Friedrich Gauss' Work in Astronomy, Number Theory, Physics and Geometry," Vol. 8 of *Rebels Who Count*, 30-minute video, Media Center, University of Toronto, 1991.

4

Transformational Geometry