## 3.3 Saccheri Quadrilaterals and Triangles

We now turn from unbounded sensed parallels and omega triangles to bounded regions, including the quadrilaterals that Saccheri used in his investigations. The Arab mathematician Omar Khayyam (circa 1050–1130) developed this quadrilateral and, in



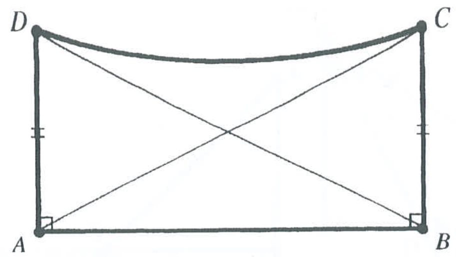


Figure 3.16

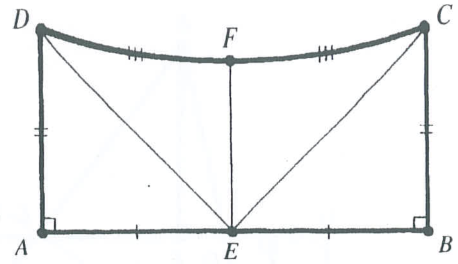


Figure 3.17

essence, Theorem 3.3.1, in his attempt to prove Euclid's fifth postulate. Saccheri is believed to have built on Khayyam's work.

**Definition 3.3.1** A *Saccheri quadrilateral* has two opposite congruent sides perpendicular to one of the other sides. The common perpendicular is the *base*, and its opposite side is the *summit*.

**Theorem 3.3.1** The summit angles of a Saccheri quadrilateral are congruent. The base and summit are perpendicular to the line on their midpoints.

***Proof.*** Let $\overline{AB}$ be the base of Saccheri quadrilateral $ABCD$ and $\overline{AC}$ and $\overline{BD}$ the diagonals (Fig. 3.16). Then $\triangle ABC \cong \triangle BAD$ by SAS, and by SSS $\triangle ADC \cong \triangle BCD$. Then the summit angles, $\angle ADC$ and $\angle BCD$, are congruent. For the second statement of the theorem, let $E$ and $F$ be the midpoints of $\overline{AB}$ and $\overline{CD}$ (Fig. 3.17). Draw $\overline{DE}$ and $\overline{CE}$. Then $\triangle DAE \cong \triangle CBE$ by SAS, and $\triangle DEF \cong \triangle CEF$ by SSS. Hence $\angle DFE \cong \angle CFE$, so $\overline{EF} \perp \overline{CD}$. Similarly, $\overline{EF} \perp \overline{AB}$. ■

**Theorem 3.3.2** The summit angles of a Saccheri quadrilateral are acute.

***Proof.*** On Saccheri quadrilateral $ABCD$ with base $\overline{AB}$, let $\overleftrightarrow{C\Omega}$ and $\overleftrightarrow{D\Omega}$ be left-sensed parallels to $\overleftrightarrow{AB}$ (Fig. 3.18). By Theorem 3.2.6, the exterior angle $\angle ED\Omega$ of omega triangle $\triangle DC\Omega$ is larger than $\angle DC\Omega$. Theorem 3.2.7 implies that the angles of parallelism $\angle AD\Omega$ and $\angle BC\Omega$ are congruent. Hence $\angle EDA$ is larger than $\angle DCB$. The summit angles are congruent, so $\angle EDA$ is bigger than $\angle CDA$. These two angles form a straight line, so the smaller, a summit angle, must be acute. ■

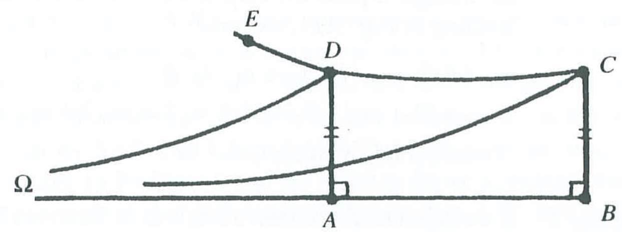


Figure 3.18

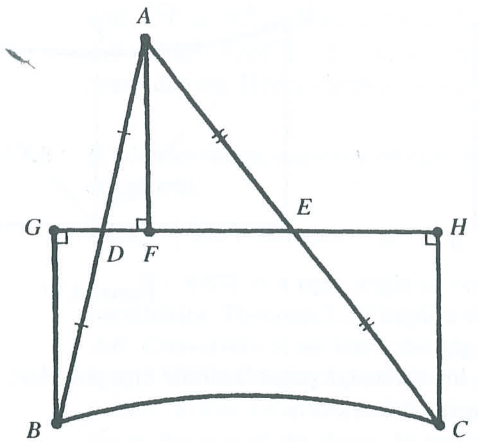


Figure 3.19

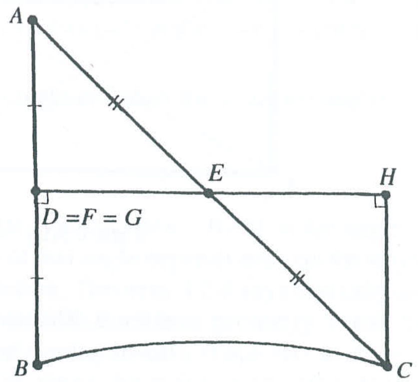


Figure 3.20

A direct consequence of Theorem 3.3.2 is that the angle sum of a Saccheri quadrilateral is always less than 360°. We convert this fact to the corresponding fact about triangles in Theorem 3.3.3 by dissecting a triangle to form a Saccheri quadrilateral (Fig. 3.19). Then we show that the sum of the measures of the summit angles of the Saccheri quadrilateral equals the angle sum of the triangle.

**Theorem 3.3.3** The angle sum of any triangle is less than 180°.

***Proof.*** In any triangle $\triangle ABC$, let $D$ be the midpoint of $\overline{AB}$ and $E$ be the midpoint of $\overline{AC}$. Construct the perpendiculars $\overline{AF}$, $\overline{BG}$, and $\overline{CH}$ to $\overleftrightarrow{DE}$.

*Claim.* Either $D$ is between $F$ and $G$ or $D = F = G$. (See Problem 3.) This claim implies that the construction looks like that shown in Fig. 3.19, 3.20, or 3.21.

*Cases 1 and 2* These cases correspond to Figs. 3.19 and 3.20. (See Problem 4.)

*Case 3* Let $D$ lie between $F$ and the other points $G$, $E$, and $H$ on this line (Fig. 3.21). In $\triangle AFD$ and $\triangle BGD$, we know that $\overline{AD} \cong \overline{BD}$, $\angle AFD \cong \angle BGD$, and $\angle ADF \cong \angle BDG$. Hence, by AAS, $\triangle AFD \cong \triangle BGD$. Similarly, $\triangle AFE \cong \triangle CHE$. Then $\overline{BG} \cong \overline{AF} \cong \overline{CH}$, which shows that $GHCB$ is a Saccheri quadrilateral. Thus both summit angles, $\angle CBG$ and $\angle BCH$, are acute. Finally we need to show that the angle sum of the triangle equals the sum of the measures of these two summit angles. Following the labeling in Fig. 3.21, we have

$$\begin{aligned} &m\angle ABC + m\angle BAC + m\angle ACB \\ &= m\angle 2 + m\angle 1 + m\angle 4 + m\angle 5 = m\angle 2 + m\angle 3 + m\angle 4 + m\angle 5 = m\angle 2 + m\angle 6 + m\angle 5 \\ &= m\angle GBC + m\angle HCB. \quad ■ \end{aligned}$$

**Exercise 1** If we dissect a Euclidean triangle as in Theorem 3.3.3, what type of quadrilateral results? Is it a Saccheri quadrilateral?



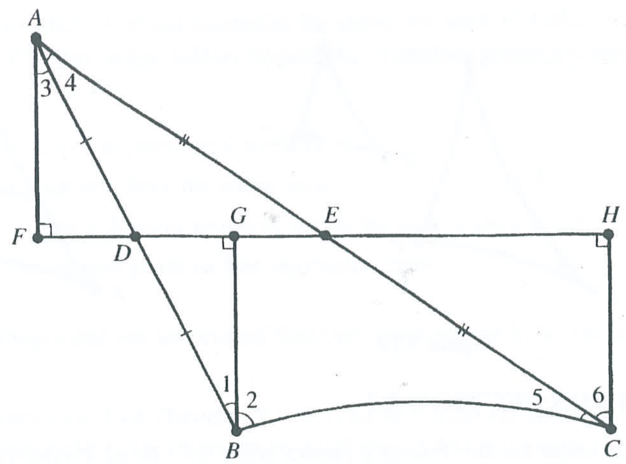


Figure 3.21

**Exercise 2** Explain why the argument in the preceding proof wouldn't show that for Euclidean triangles the sum of the measures of their angles is less than 180°.

**Corollary 3.3.1** The angle sum of a quadrilateral is less than 360°.

***Proof.*** In any quadrilateral $ABCD$ draw $\overline{BD}$. If $ABCD$ is the union of $\triangle ABD$ and $\triangle CBD$, we can use Theorem 3.3.3 twice. Otherwise, $A$ and $C$ are on the same side of $\overline{BD}$. The definition of a polygon (Problem 4 of Section 1.2) implies that $A$ is in the interior of $\triangle CBD$ or that $C$ is in the interior of $\triangle ABD$. Either way $ABCD$ is the union of $\triangle ABC$ and $\triangle ADC$. Now use Theorem 3.3.3 twice. In Problem 5 you are asked to fill out this argument. ■

The next theorem shows that the powerful Euclidean notion of similar triangles plays no role in hyperbolic geometry. In hyperbolic geometry similar triangles are always congruent. Indeed, as Saccheri noted, the existence of noncongruent but similar triangles is equivalent to Playfair's axiom.

**Theorem 3.3.4** If two triangles have corresponding angles congruent, then the triangles are congruent.

***Proof.*** Let $\triangle ABC$ and $\triangle A'B'C'$ have corresponding angles congruent (Fig. 3.22). If one pair of corresponding sides is congruent, then by ASA the triangles would be congruent. For a contradiction, assume WLOG that $\overline{AB}$ is longer than $\overline{A'B'}$. Construct $D$ between $A$ and $B$ such that $\overline{AD} \cong \overline{A'B'}$ and construct $E$ on the ray $\overrightarrow{AC}$ so that $\overline{AE} \cong \overline{A'C'}$. Then by SAS, $\triangle ADE \cong \triangle A'B'C'$, which gives the two cases shown in Fig. 3.23 to consider. In Problem 6 you are asked to derive a contradiction in each case. These contradictions show that $\overline{AB}$ cannot be longer than $\overline{A'B'}$, and so the triangles are congruent. ■

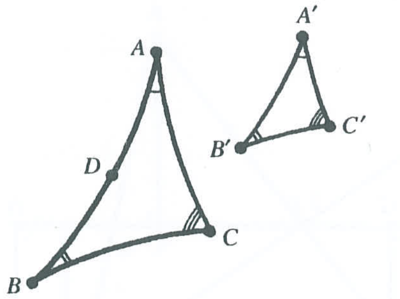


Figure 3.22

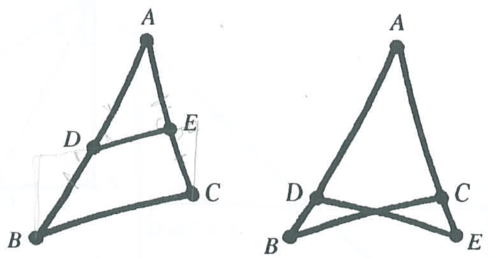


Figure 3.23

## PROBLEMS FOR SECTION 3.3

1. Use the unit circle for the Poincaré model and construct a Saccheri quadrilateral as follows. Let the base be on $y = 0$, the two sides be on $[x \pm (5/3)]^2 + y^2 = (4/3)^2$ and the summit be on $x^2 + [y - (5/3)]^2 = (4/3)^2$. Graph these four hyperbolic lines. Explain why the two sides are perpendicular to the base. Why can you reasonably expect the sides to be congruent? Verify visually that the summit angles are acute.
2. **a)** Prove that the supposition in Problem 7 of Section 3.2 is correct. [*Hint:* Consider $X$ on $\overrightarrow{DC}$ such that $ABDX$ is a Saccheri quadrilateral. Use Theorem 3.3.2.]

   **b)** Explain why the supposition you showed in part (a) implies, as Saccheri proved, that sensed parallels approach one another.
3. (Proof of the claim in Theorem 3.3.3.) If $D$, $F$, and $G$ are the same point or $D$ is between $F$ and $G$, the claim holds. Suppose, for a contradiction, some other situation occurs. That is, either $D$ coincides with just one of the other points or that $F$ and $G$ are both on the same side of $D$. Find a contradiction for each of these options. [*Hint:* Use Euclid I-16 for the second option.]
4. Prove the first two cases of Theorem 3.3.3, based on Figs. 3.19 and 3.20.
5. Illustrate and complete the proof of Corollary 3.3.1.
6. **a)** Finish the proof of Theorem 3.3.4. [*Hint:* Use Corollary 3.3.1 and Euclid I-16.]

   **b)** Explain why part (a) doesn't apply in Euclidean geometry.
7. As shown in Fig. 3.24, $\triangle ABC$ and $\triangle ABD$ have angle sums less than 180°. Which triangle, if any, has the larger sum? Prove your answer.

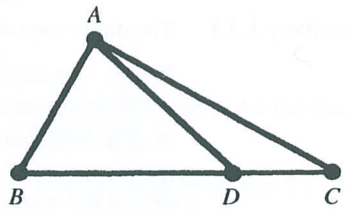


Figure 3.24

8. Generalize Corollary 3.3.1 to convex polygons with $n$ sides. Prove your generalization by using induction. Does this generalization hold if the polygon isn't convex? Explain.
9. In Section 3.4 we will define the *defect* of a triangle to be the difference between 180° and the angle sum of the triangle. Prove that the defects of triangles are additive in the following sense. If $S$ lies between $Q$ and $R$ in $\triangle PQR$, show that the defect of $\triangle PQR$ equals the sum of the defects of $\triangle PQS$ and $\triangle PSR$.

## 3.4 Area and Hyperbolic Designs

Areas in hyperbolic geometry do not have easily remembered formulas, such as $A = \frac{1}{2}bh$ for a Euclidean triangle. In place of that, Theorem 3.4.5 asserts that the area of a triangle is proportional to the *defect* of the triangle, or the amount by which its angle