3.2 Properties of Lines and Omega Triangles

We develop hyperbolic geometry following the path of Saccheri, Gauss, Lobachevsky, and Bolyai, but adding the foundation of Hilbert and others. We can thus accentuate the geometric intuition of earlier work while having the logical basis required to avoid the flaws discovered later.

Our undefined terms are *point, line, on, between,* and *congruent*. Our axioms are Hilbert's axioms for plane geometry (see Appendix B), with the exception of axiom IV-1; Playfair's axiom, which we replace with the characteristic axiom of hyperbolic geometry. (See Section 3.1.) We also avail ourselves of Euclid's first 28 propositions, or those he proved without using the parallel postulate. (See Appendix A.) These propositions give us many of the key properties about lines and triangles, which we need in our development of hyperbolic geometry. They are provable in Hilbert's axiomatic system without our having to use either axiom IV-1 or the characteristic axiom.

Like the developers of hyperbolic geometry, we prove many theorems in this chapter, including Theorem 3.2.1, by contradiction. Until 1868, when Beltrami built the first model of hyperbolic geometry, mathematicians wondered whether the characteris-

tic axiom itself was potentially a contradiction. A rigorous proof of even Theorem 3.2.1 requires considerable development from Hilbert's axioms. For example, in Fig. 3.8 we would need to show that B can always be chosen so that the line m enters $\triangle ABP$ at P. (By *enter* we mean that the line has P and another point inside of the triangle on it. (See Moise [8, Chapter 24] for a careful development of these basics, including the justification of right- and left-sensed parallels in Definition 3.2.1.)

Theorem 3.2.1 Given a point P not on a line k, there are infinitely many lines on P that have no points also on k.

Proof. Let the two lines indicated by the characteristic axiom be l and m. Pick points A on k and B and C on l such that line m enters $\triangle ABC$ at P. By Pasch's axiom, m must also intersect the triangle on another side, WLOG at D on $\overline{AB}$. Now, for every point X on the segment $\overline{BD}$, we can draw $\overleftrightarrow{PX}$. (See Fig. 3.8.)

Claim. $\overleftrightarrow{PX}$ does not intersect k. Suppose, for a contradiction, that $\overleftrightarrow{PX}$ and k had the point Y in common. Then line m would enter triangle $\triangle XYA$ at D, and by Pasch's axiom m would have to intersect the triangle again on $\overline{XY}$ or on $\overline{YA}$. However, m already intersects $\overleftrightarrow{PX}$ at P, eliminating $\overline{XY}$. Furthermore $\overline{YA}$ is part of k, which has no point in common with m, which is a contradiction. Hence each of the infinitely many lines $\overleftrightarrow{PX}$ has no point in common with k. ■

For the same k and P as in the preceding proof, the lines through P split into two categories: those that intersect k and those that do not. In Fig. 3.8, for Z on segment $\overline{AD}$, some of the lines $\overleftrightarrow{PZ}$, such as $\overleftrightarrow{PA}$, intersect k, but others such as $\overleftrightarrow{PD}$, don't. By the continuity of a line, some point W must separate these lines. Does $\overleftrightarrow{PW}$ intersect k? Draw a figure illustrating the following argument by contradiction showing that $\overleftrightarrow{PW}$ cannot intersect k. If T is the supposed intersection of $\overleftrightarrow{PW}$ and k, consider any point S on k with T between A and S. Now draw $\overleftrightarrow{PS}$, which clearly intersects k at S. Therefore $\overleftrightarrow{PW}$ would not be the last line, contradicting the assumption about W. Explain why a similar situation occurs on the left side of Fig. 3.8.

Definition 3.2.1 Given a point P not on line k, the first line on P in each direction that does not intersect k is the (*right-* or *left-*) *sensed parallel* to k at P. Other lines on P that do not intersect k are called *ultraparallel* to k. Let A be on k with $\overline{AP} \perp k$. Call the smaller of the angles

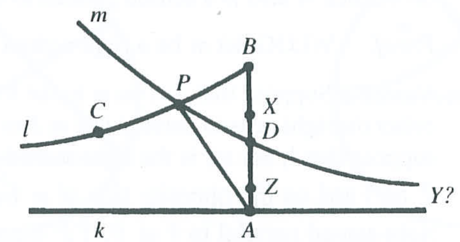

Figure 3.8

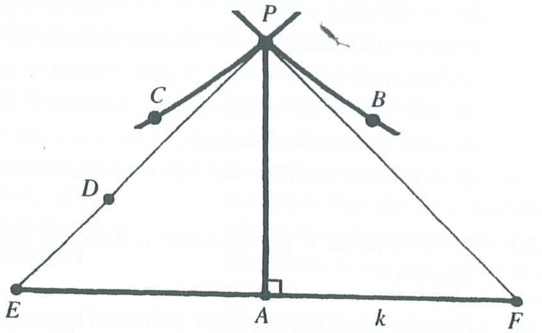

Figure 3.9

a sensed parallel makes with $\overline{AP}$ the *angle of parallelism at* P. (If the angles are equal, either is the angle of parallelism.)

One goal of this section is to show that the angle of parallelism depends only on the length of segment $\overline{AP}$.

Theorem 3.2.2 If l and m are the two sensed parallels to k at P, they have the same angle of parallelism.

Proof. As shown in Fig. 3.9, let $\overline{AP} \perp k$ and $\overleftrightarrow{PB}$ and $\overleftrightarrow{PC}$ be the sensed parallels to k at P. For a contradiction, let the angles of parallelism differ, say, $m\angle APB < m\angle APC$. Construct inside $\angle APC$ a new angle $\angle APD \cong \angle APB$. Because $\overleftrightarrow{PC}$ is a sensed parallel, $\overleftrightarrow{PD}$ intersects k, say, at E, giving a triangle $\triangle APE$. Let F be on k with $\overline{AF} \cong \overline{AE}$. Then $\triangle APF \cong \triangle APE$ by SAS (Euclid I-4). But $\angle APF \cong \angle APB$, which means that $\overleftrightarrow{PF}$ and $\overleftrightarrow{PB}$ are the same line by Hilbert's axiom III-4. However, $\overleftrightarrow{PB}$ is a sensed parallel to k, so it cannot intersect k at F, which is a contradiction. Hence the angles of parallelism must be the same. ■

Corollary 3.2.1 All angles of parallelism are less than a right angle. Two lines with a common perpendicular are ultraparallel.

Proof. See Problem 2. ■

Theorem 3.2.3 Let P be a point not on k and let m be a sensed parallel to k at P. If S is any other point on m, then m also is a sensed parallel to k at S.

Proof. WLOG, let m be a right-sensed parallel to k at P.

Case 1 Suppose that S is on m to the left of P. Because m and k do not intersect, m is either the right-sensed parallel to k at S or m is ultraparallel to k at S. For a contradiction, suppose that l, not m, is the right sensed parallel to k at S. As shown in Fig. 3.10, pick T on l and on the opposite side of m from k. Then $\overleftrightarrow{TP}$ crosses m at P. As m is the right-sensed parallel to k at P, $\overleftrightarrow{TP}$ must intersect k, say, at A. Then $\overline{SA}$ lies below l because l is a right-sensed parallel. Let U be on m with S between P and U. Then $\overline{UA}$ is below $\overline{SA}$. Now l enters $\triangle UPB$ at S and, by Pasch's axiom, l would need to intersect either

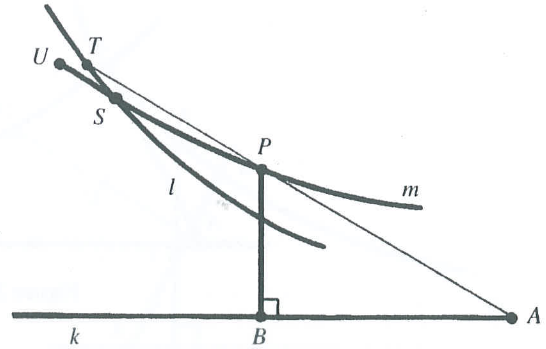

Figure 3.10

$\overline{UB}$ or $\overline{PB}$. If l intersects $\overline{UB}$, it can't be right sensed parallel to k, a contradiction. But if l intersects $\overline{PB}$, it enters $\triangle PBA$ on that side. By Pasch's axiom, either l intersects $\overline{BA}$, which is k, or it intersects $\overline{PA}$. However, l already intersected $\overline{PA}$ at T, so that is impossible. Finally, if l intersects k, it isn't right sensed parallel. Hence m is the right-sensed parallel to k at S.

Case 2 Let S be on m to the right of P. (See Problem 3.) ■

3.2.1 Omega triangles

In both the Klein and Poincaré models of hyperbolic geometry (Fig. 3.11) sensed parallels "meet" on the circular boundary. Following the historical development, we say that sensed parallels meet at imaginary points called *omega points*. Although the originators of hyperbolic geometry didn't have any models, they benefited greatly from thinking of sensed parallels "meeting at infinity." In particular, they used *omega triangles*, which have one omega point, to prove theorems about ordinary triangles. They first showed, as do we here, that omega triangles share some key properties with regular triangles. In the Klein and Poincaré models, omega triangles (Definition 3.2.2) are triangles with one vertex on the boundary. These models also suggest Theorem 3.2.4, which we accept without proof. (For a proof, see Moise [8, 321–322].)

Definition 3.2.2 All lines right- (left-) sensed parallel to a given line are said to have the same *right (left) omega point*. An *omega triangle* $\triangle AB\Omega$ consists of two (ordinary) points A and B, the segment $\overline{AB}$, and the sensed parallel rays $\overrightarrow{A\Omega}$ and $\overrightarrow{B\Omega}$.

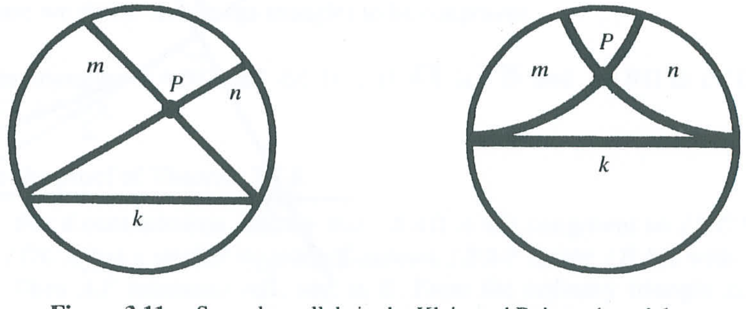

Figure 3.11 Sensed parallels in the Klein and Poincaré models.

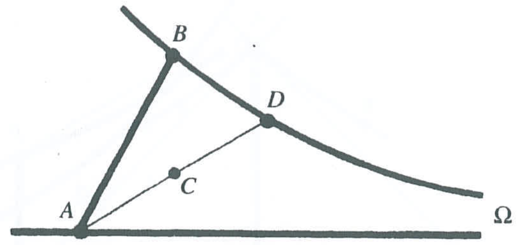

Figure 3.12

Theorem 3.2.4 If m is sensed parallel to k, then k is sensed parallel to m. If m is sensed parallel to l and l is sensed parallel to k with the same omega point, then m is sensed parallel to k with the same omega point.

Exercise 1 Draw an omega triangle for the Klein model.

Theorem 3.2.5 **Modified Pasch's Axiom for Omega Triangles** If a line k contains a point interior to $\triangle AB\Omega$ and k is on one of the vertices, then k intersects the opposite side of $\triangle AB\Omega$.

Proof. Let C be in $\triangle AB\Omega$, and draw line $\overleftrightarrow{AC}$ (Fig. 3.12). Because $\overleftrightarrow{A\Omega}$ is the sensed parallel to $\overleftrightarrow{B\Omega}$ at A, for $k = \overleftrightarrow{AC}$, k intersects $\overleftrightarrow{B\Omega}$, say, at D. The line $\overleftrightarrow{BC}$ is handled similarly. We can extend this theorem by treating Ω as a vertex. The line $\overleftrightarrow{C\Omega}$ enters the ordinary triangle $\triangle ABD$ at C, and by Pasch's axiom $\overleftrightarrow{C\Omega}$ must intersect $\overline{AB}$ or $\overline{BD}$. However, if $\overleftrightarrow{C\Omega}$ intersected $\overline{BD}$, there would be two sensed parallels to $\overleftrightarrow{A\Omega}$ at that point: $\overleftrightarrow{B\Omega}$ and $\overleftrightarrow{C\Omega}$. So $\overleftrightarrow{C\Omega}$ intersects $\overline{AB}$. ■

Theorem 3.2.6 **Euclid I-16 for Omega Triangles** The measure of an exterior angle of an omega triangle is greater than the measure of the opposite interior angle.

Proof. Let $\triangle AB\Omega$ be an omega triangle and extend $\overline{AB}$ (Fig. 3.13). We prove $\angle CA\Omega$ to be greater than $\angle AB\Omega$ by showing that the other two possibilities lead to contradictions. For case 1, let $m\angle CA\Omega < m\angle AB\Omega$. Construct $\angle ABZ$ inside $\angle AB\Omega$ with

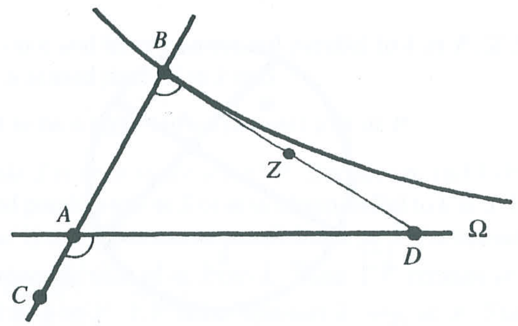

Figure 3.13 Case 1 of Theorem 3.2.6.

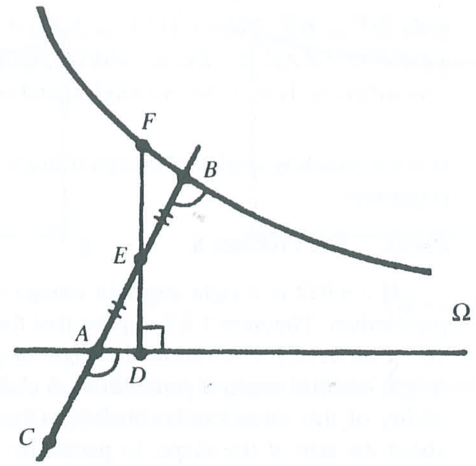

Figure 3.14 Case 2 of Theorem 3.2.6.

$\angle ABZ \cong \angle CA\Omega$. Then $\overrightarrow{BZ}$ intersects $\overrightarrow{A\Omega}$, say, at D because $\overleftrightarrow{B\Omega}$ is the sensed parallel to $\overleftrightarrow{A\Omega}$ at B. But then $\angle CA\Omega$ is an exterior angle to an ordinary triangle, $\triangle ABD$, with an opposite interior angle $\angle ABD \cong \angle CA\Omega$. This result contradicts Euclid I-16.

For case 2 (Fig. 3.14), we try to set $\angle CA\Omega \cong \angle AB\Omega$. If we let E be the midpoint of $\overline{AB}$, we can draw the perpendicular $\overline{DE}$ to $\overleftrightarrow{A\Omega}$. (The angle of parallelism is acute, so D is not A.) Construct F on $\overleftrightarrow{B\Omega}$ so that $\overline{FB} \cong \overline{AD}$ and F and D are on opposite sides of $\overleftrightarrow{AB}$. Note that $\angle FBE \cong \angle DAE$ by our extra assumption in this case. So by SAS $\triangle FBE \cong \triangle DAE$. Then the angles at E are vertical angles, ensuring that D, E, and F are on the same line, by Euclid I-14. Because $\angle ADE$ is a right angle, $\angle BFE$ is also. But then the angle of parallelism for $\overleftrightarrow{F\Omega}$ to $\overleftrightarrow{D\Omega}$ would be a right angle, which is a contradiction. Hence the assumption that $\angle CA\Omega \cong \angle AB\Omega$ must be wrong. Both alternatives are impossible, proving the theorem. ■

We can extend Euclid's concept of congruent triangles to omega triangles. However, the lengths of two of the "sides" of an omega triangle are infinite, and we can hardly measure the imaginary angle "at" the omega point. Hence there are only two angles and the included side to consider in each omega triangle. We show that if two omega triangles have two of these three parts congruent, they have their third parts congruent. In this case we define the omega triangles to be *congruent*.

Theorem 3.2.7 In omega triangles $\triangle AB\Omega$ and $\triangle CD\Lambda$, if $\overline{AB} \cong \overline{CD}$ and $\angle AB\Omega \cong \angle CD\Lambda$, then $\triangle AB\Omega \cong \triangle CD\Lambda$.

Exercise 2 Illustrate the proof of Theorem 3.2.7.

Proof. For a contradiction, assume that $\angle BA\Omega$ is not congruent to $\angle DC\Lambda$, and so WLOG $\angle DC\Lambda$ has a smaller measure. Construct $\angle BAP$ inside $\angle BA\Omega$ with $\angle BAP \cong \angle DC\Lambda$. Then $\overleftrightarrow{AP}$ intersects $\overrightarrow{B\Omega}$, say, at E. From the ordinary triangle $\triangle ABE$ we construct a congruent one in the other omega triangle. Let F be the point on $\overrightarrow{D\Lambda}$

with $\overline{DF} \cong \overline{BE}$. Then $\triangle DFC \cong \triangle BEA$ by SAS. However, this outcome means that $\angle DCF \cong \angle BAE \cong \angle DC\Lambda$, which implies that $\overleftrightarrow{C\Lambda}$ would intersect $\overleftrightarrow{D\Lambda}$, which is a contradiction. Hence the two angles, and so the two omega triangles, are congruent. ■

Theorem 3.2.8 If corresponding angles of omega triangles are congruent, then the omega triangles are congruent.

Proof. See Problem 8. ■

If $\angle AB\Omega$ is a right angle in omega triangle $\triangle AB\Omega$, then $\angle BA\Omega$ is the angle of parallelism. Theorem 3.2.7 implies that the size of this angle depends only on the length AB. Conversely, if we know the angle of parallelism, Theorem 3.2.8 says that only one length has that angle of parallelism. A comparison with Euclidean geometry reveals the oddity of this situation. In Euclidean geometry the angles of a shape tell us nothing about the size of the shape. In particular, similar shapes have the same angles but are of different sizes. In hyperbolic geometry, the angles of shapes determine their size. In both geometries we can measure angles absolutely by comparing them with a right angle. However, Theorem 3.2.8 says that in hyperbolic geometry we can, in principle, measure lengths in an absolute way by using the angle of parallelism.

PROBLEMS FOR SECTION 3.2

1. Suppose that m is a left-sensed parallel to k and l is a right-sensed parallel to k. Use the Klein model to determine whether m and l can (a) intersect, (b) be sensed parallel, or (c) be ultraparallel.
2. Prove Corollary 3.2.1.
3. Prove case 2 of Theorem 3.2.3 [*Hint:* Pick T on l between m and k.]
4. Relate Euclid I-27 and I-28 to Corollary 3.2.1.
5. Prove Pasch's axiom (Hilbert II-4) for omega triangles.
6. Suppose that $\overleftrightarrow{AC} \perp \overleftrightarrow{C\Omega}$, B is between A and C and that $\overleftrightarrow{A\Omega}$, $\overleftrightarrow{B\Omega}$, and $\overleftrightarrow{C\Omega}$ are all right-sensed parallels. Theorem 3.2.8 implies that the angles of parallelism for $\triangle AC\Omega$ and $\triangle BC\Omega$ cannot be equal. Which is larger? Prove your answer.
7. Let $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ be sensed parallels, $\overline{AB} \perp \overleftrightarrow{BD}$ and $\overline{CD} \perp \overleftrightarrow{BD}$. Suppose, as Fig. 3.15 suggests, that $\overline{CD}$ is shorter than $\overline{AB}$. Use your answer to Problem 6 to compare 360° with the angle sum of the quadrilateral $ABCD$.

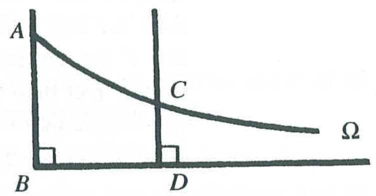

Figure 3.15

8. Prove Theorem 3.2.8.
9. If M is the midpoint of $\overline{AB}$ in $\triangle AB\Omega$, prove that $\angle A \cong \angle B$ iff $\angle AM\Omega$ is a right angle.
10. In omega triangles $\triangle AB\Omega$ and $\triangle CD\Lambda$, if $\angle A \cong \angle B$, $\angle C \cong \angle D$, and $\overline{AB} \cong \overline{CD}$, prove that $\triangle AB\Omega \cong \triangle CD\Lambda$.

3.3 Saccheri Quadrilaterals and Triangles

We now turn from unbounded sensed parallels and omega triangles to bounded regions, including the quadrilaterals that Saccheri used in his investigations. The Arab mathematician Omar Khayyam (circa 1050–1130) developed this quadrilateral and, in