3

Non-Euclidean Geometries

The repeated insects in this design clearly follow some rule, although the geometry behind this pattern may seem mysterious. Douglas Dunham, a geometer with an interest in computer graphics, has combined his knowledge of non-Euclidean geometries and computers to produce a large variety of such designs. The fundamentals of non-Euclidean geometries are basic to an understanding of the mathematics of these designs.

> The most suggestive and notable achievement of the last century is the discovery of non-Euclidean geometry. —*David Hilbert*

> To this interpretation of geometry I attach great importance for should I have not been acquainted with it, I never would have been able to develop the theory of relativity. —*Albert Einstein*

3.1 Overview and History

The classical understanding of axioms (postulates) as "self-evident truths" was shattered in mathematics by the introduction and development of non-Euclidean geometries during the nineteenth century. The story of non-Euclidean geometries plays an important role in the history of ideas, and mathematics students deserve to know it. In addition, non-Euclidean geometries illustrate the need to transcend the intuitive models of elementary mathematics, allowing us to think successfully about the much more abstract concepts of modern mathematics.

In this chapter we focus primarily on the specific non-Euclidean geometry now called *hyperbolic geometry*. This focus provides an in-depth look at an axiomatic system. In the final section of this chapter we consider two other non-Euclidean geometries: spherical and single elliptic geometries.

Until the nineteenth century, no one questioned the truth of Euclidean geometry, although many sought to remove a perceived blemish in Euclid's masterpiece. His fifth postulate about parallels was hardly a "self-evident truth." (See Section 1.2.) Mathematicians tried repeatedly to prove this postulate from the others. For the most part, though, they either explicitly or implicitly used equivalent assumptions, such as Playfair's, which is now used in geometry books. (Recall that Playfair's axiom states: "Through a given point not on a given line m there passes at most one line which does not intersect m.") Before 1800, the person who came closest to realizing that the fifth postulate could not be proved from the others was the Italian mathematician Girolamo Saccheri (1667–1733). His approach was to start from two negations of the fifth postulate and look for a contradiction. From one he found the desired contradiction. From the one leading to hyperbolic geometry he deduced increasingly bizarre consequences, such as the angle sum of a triangle is less than 180° and the existence of straight lines that approach each other but never cross. However, he found no explicit contradiction. Finally, he concluded that "the hypothesis . . . is absolutely false, because it is repugnant to the nature of the straight line." Bonola [1,43] Saccheri could receive credit for logically developing a non-Euclidean geometry, but he could as easily be seen as remaining inside the world view of asserting Euclidean geometry to be true and to be the geometry of the physical world.

That world view, accepted for centuries, found even more support in the 1700s. First, Immanuel Kant (1724–1804), the most influential philosopher of the time, held that the truths of mathematics differed fundamentally from incidental facts about the world, such as the earth has one moon. Mathematics, Kant taught, was *a priori* true—necessarily true, even before any particular experience we have. Geometry, as developed by Euclid, seemed a compelling example of how humans could obtain absolute knowledge about the world. In addition Kant argued that we needed an essentially inborn understanding of geometry and space before we could experience anything in space.[1] He thought that geometry had to be Euclidean.

In the eighteenth century, the Age of Enlightenment, mathematicians and philosophers built on the perceived absolute truth of mathematics in a second way. The astounding success of Newton's calculus and physics convinced his successors that mathematics wasn't just true in some metaphysical sense, but also in a tangible sense. The ideal world of mathematics, it seemed, was the real world. The physical meaning of much of the mathematics developed in the eighteenth century was so convincing that the rigorous deductive methods of Greek geometry seemed superfluous. The shock of the radically different mathematical results of the nineteenth century, starting with non-Euclidean geometries, forced mathematicians to reintroduce careful proofs.

The first person to break from the world view of Euclidean geometry, its unquestionable truth, and applicability to the physical world was Carl Friedrich Gauss (1777–1855). Despite his fame, Gauss never published anything on non-Euclidean geometries because he feared ridicule. Nicolai Lobachevsky (1793–1856) and János Bolyai (1802–1860), the two young mathematicians who did publish on non-Euclidean geometry, were greeted with silence for years after their publications in 1829 and 1832, respectively. Indeed, only with the publication of Gauss's notes after his death did the wider community of mathematicians start investigating non-Euclidean geometries.

3.1.1 The advent of hyperbolic geometry

Hyperbolic geometry, the non-Euclidean geometry that Gauss, Lobachevsky, and Bolyai developed, retains Euclid's first four postulates and changes the fifth postulate to the following axiom. In Sections 3.2, 3.3, and 3.4, we investigate hyperbolic geometry in much the same way that these mathematicians did. However, to make the process clearer, we make explicit certain logically necessary assumptions that had been overlooked until the end of the nineteenth century. (See Section 1.3.) Hyperbolic geometry is sometimes called Lobachevskian geometry to honor Lobachevsky's priority in publishing. Felix Klein called this geometry *hyperbolic* in his classification of geometries, which we discuss in Section 6.5.

Characteristic Axiom of Hyperbolic Geometry Given any line k and any point P not on k, there are at least two lines on P which do not intersect k (Fig. 3.1).

Various consequences follow from this change, including the many that Saccheri found. The most startling is the theorem that the measures of the angles of a triangle do not add up to 180°, as they do in Euclidean geometry. The goal in our study of hyperbolic geometry, Theorem 3.1.1, goes even further, relating the angle sum to the area of the triangle (Fig. 3.2). The greater the area of the triangle is, the smaller the angle sum is. Consequently, any triangle has a maximum area. As the sides of triangles can become indefinitely long, this consequence seems paradoxical.

[1] Studies of blind people and people with recovered sight indicate that Kant's argument is incorrect; sight is essential to developing the usual conception of space. (See Sacks [11, 124].)

NIKOLAI LOBACHEVSKY AND JÁNOS BOLYAI

Nikolai Lobachevsky (1793–1856) was a mathematics professor at the University of Kazan in Russia, where he first published on hyperbolic geometry in 1829. His extensive development included its trigonometry, corresponding to the trigonometry on a sphere of imaginary radius. He thought that this geometry might be pertinent to the study of astronomy because he realized that, as distances increased, the difference between Euclidean geometry and hyperbolic geometry became more noticeable. In his publications, Lobachevsky expounded on this geometry.

János Bolyai (1802–1860) was a Hungarian army officer with a reputation for dueling. One day he reportedly dueled several officers, playing his violin between duels. Bolyai found basically the same results as Lobachevsky. His publication of 1832, *The Science of Absolute Space*, also reported his investigation of properties common to both Euclidean and hyperbolic geometries.

The fame of Lobachevsky and Bolyai rests entirely on their work in hyperbolic geometry. Surprisingly the wider mathematical community neglected their publications, which deeply disappointed both men. Certainly, the original languages of Russian and Hungarian deterred readers. However, in 1840 Lobachevsky published a book on his research in German, then the foremost language of mathematics. Perhaps more significantly, geometric research focused then on projective geometry, which does not study parallel lines. Also, the trigonometric formulations may have impeded mathematicians interested in philosophical and geometric implications. Only after the posthumous publication of Gauss's notebooks in 1855 did an active study of non-Euclidean geometry begin. Because of the number of distinguished mathematicians who had failed to prove Euclid's fifth postulate, Lobachevsky and Bolyai should have found an interested, critical audience for their radical answer to this fundamental question in geometry.

Theorem 3.1.1 In hyperbolic geometry the difference, $180° - (m\angle A + m\angle B + m\angle C)$, between 180° and the angle sum of a triangle is proportional to the area of the triangle.

Exercise 1 Compare Theorem 3.1.1 with Theorem 1.6.3.

Theorem 3.1.1 suggests that we could decide "which geometry is true" by measuring real triangles. Since 1890, mathematicians and physicists have realized the futility

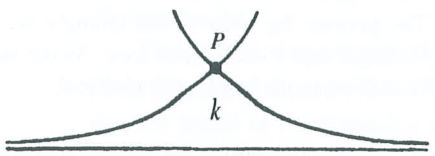

Figure 3.1

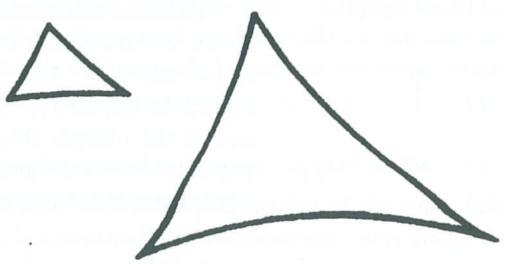

Figure 3.2

CARL FRIEDRICH GAUSS

Carl Friedrich Gauss (1777–1855) dominated mathematics during the first half of the nineteenth century, making fundamental contributions to virtually every area of mathematics. The prince of his small German state sponsored his education after learning of his prodigious abilities as a child. By age five he had found an arithmetic error in his father's accounts. In another story his grade school teacher made the class add the numbers from one to one hundred. After some thought Gauss found a formula and simply wrote down the correct answer, whereas his classmates toiled and made mistakes.

Gauss soon won acclaim as a mathematician. At 18 he developed the method of least squares, which is used extensively in statistics. A year later he constructed with straightedge and compass a regular 17-sided polygon and later characterized all constructible regular polygons, the first such advances since the ancient Greeks. He earned his Ph.D. at 22 by giving the first proof of the fundamental theorem of algebra, which says that every polynomial with real or complex coefficients has all of its roots in the complexes. Two years later, he published his first major treatise on number theory, rejuvenating that ancient area. That same year, 1801, Gauss astonished astronomers by determining the orbit of the first asteroid, Ceres, based on only a few observations and after it had been lost from view owing to weather conditions. To do so he had to generalize his method of least squares. Gauss subsequently became a professor of astronomy and director of the observatory in Göttingen.

By 1800 Gauss had become convinced that Euclid's parallel postulate could not be proven, and in installments he developed what he called non-Euclidean geometry. Although none of this work was published until after his death, he did correspond with a number of mathematicians about it. Gauss also made seminal contributions to differential geometry, including the curvature of a surface and geodesics. He showed that Gaussian curvature determined all the properties of a surface not related to how it is placed in a surrounding space.

Gauss's contributions on complex numbers persuaded mathematicians that complex numbers were essential in mathematics. He represented complex numbers geometrically, extended number theory to complex integers, and contributed fundamentally to what is now called complex analysis. He advanced knowledge in astronomy, magnetism, optics, and other applied fields.

Gauss's contemporaries revered him as the "Prince of Mathematicians," and he is often considered the greatest mathematician since Newton. He was also one of the last who could claim to know all the mathematics in existence at his time. Gauss's work can be seen both as crowning the great expansion of mathematics in the sixteenth and seventeenth centuries and as igniting the explosion of specialized, abstract mathematics since then.

of empirically deciding which geometry is correct because physical assumptions to test mathematical relations must be made. For example, we would have to assume that the path of a light ray is a straight line in order to measure even a moderately large triangle.

The first notable response to the advent of hyperbolic geometry came in 1854 in a lecture delivered by Georg Friedrich Bernhard Riemann (1826–1866) for his introductory lecture to the faculty at Göttingen University. Only Riemann's teacher, the aging

This paragraph needs to be rewritten. There is no evidence that Riemann knew about hyperbolic geometry at the time of his 1854 talk.

GEORG FRIEDRICH BERNHARD RIEMANN

Bernhard Riemann (1826–1866) had obvious mathematical ability early in his life. Nevertheless, he started studying theology at age 20 at the request of his father, a Lutheran minister. Within a year, though, he had turned to mathematics and finished his Ph.D. at age 25 under Gauss's direction. He became a professor at Göttingen University in Germany when he was 27 and remained there until his health deteriorated. Riemann suffered from tuberculosis for nearly the last four years of his life, working when he was well enough and dying before the age of 40.

Although Riemann is also remembered for Riemann sums and integrals that appear in calculus texts, his major work focused on physics and more advanced mathematics. Riemann made profound contributions to complex analysis and what later became topology and differential geometry. He blended deep physical and geometric intuition with insightful arguments. Some of his contemporaries criticized his proofs for a lack of rigor. However, his approaches, conjectures, and results have shaped all the areas he investigated.

Barely 10 years after mathematicians started exploring n-dimensional Euclidean geometry, Riemann's introductory lecture as a professor in 1854 developed a concept of space far more general. His vision of what we now call differential geometry included Euclidean, hyperbolic, spherical, and elliptic geometries, in any number of dimensions, as special cases. He showed how to base the concept of space on theoretical or physical measurements of distances. How those measurements of a space differed from the corresponding Euclidean measurements based on the Pythagorean theorem described how that space was curved. Sixty years after Riemann's death his work became the foundation for Einstein's general theory of relativity, which related those measurements to the effects of gravity.

Gauss, apparently caught the point of this lecture, entitled "On the Hypotheses which underlie Geometry." However, this talk, published after Riemann's death, focused geometric thought on a new field, differential geometry, and spurred an active debate on non-Euclidean geometries. Riemann had realized that the work of Gauss, Lobachevsky, and Bolyai was more than playing abstractly with a postulate. In essence, he recognized that the revised postulate implied that space had to be shaped differently than what Euclid's fifth postulate implied. He then articulated how infinitely many different geometries could be created, each with its own "shape."

Differential geometry, the field pioneered by Riemann, Gauss, and others, investigates geometries by looking at how they behave in small regions and, in particular, how they curve. Where Euclidean geometry is flat and spherical geometry is curved positively, hyperbolic geometry has a uniform negative curvature, as the model of the pseudosphere (Fig. 3.3) indicates. Riemann envisioned geometries in any number of dimensions with changing curvatures throughout. The general theory of relativity builds on Riemann's work on the curvature of space. Einstein used a four-dimensional geometry (three spatial dimensions and time) curved at each point by the gravitational forces acting there. Light waves travel along *geodesics*, paths of shortest length following the curvature of the surrounding space. In a sense, Einstein's theory settled the nineteenth

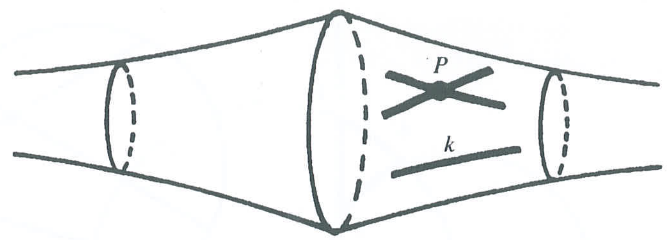

Figure 3.3 The effect of the characteristic axiom: A pseudosphere.

century question of the empirical truth of Euclidean and hyperbolic geometries in a surprising way: Both are false! (See do Carmo [3], Kline [6], McCleary [7], Rucker [10], and Weeks [13] for further information.)

The abstraction throughout mathematics and the strangeness of the new geometries led mathematicians to search again for absolute certainty in their mathematical arguments. No longer could mathematicians rely on an intuitive model to reveal the essential idea behind an argument. They examined the axioms for geometry, culminating in Hilbert's axioms for Euclidean geometry (see Section 1.3). Hilbert chose these axioms so that, by replacing the axiom for parallel lines with the characteristic axiom that we just presented, he would have axioms for hyperbolic geometry. (See Bonola [1] and Kline [6, Chapter 36] for further historical information.)

3.1.2 Models of hyperbolic geometry

In the latter part of the nineteenth century, various mathematicians developed models of hyperbolic geometry. We partially treat some of these models here. Each has some disadvantages, but they all help give a feel for this geometry. These models are based in part on a Euclidean plane or space, with suitable interpretations of the undefined terms. For the Poincaré model, we first need a definition: Two (Euclidean) circles are *orthogonal* iff the radii of these circles at the circles' points of intersection are perpendicular (Fig. 3.4). In all of these models the terms *on* and *between* have their usual meaning. In the language of Section 1.4, these models show the relative consistency of hyperbolic geometry based on Euclidean geometry.

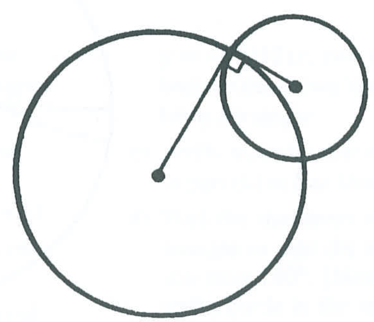

Figure 3.4 Orthogonal circles.

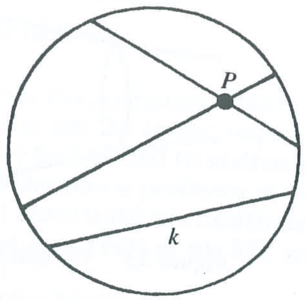

Figure 3.5 The effect of the characteristic axiom: The Klein model.

The Klein Model (1871). *Point* means a point in the interior of a particular Euclidean circle. *Line* means the portion of a Euclidean line in the interior of that circle. Figure 3.5 readily illustrates that the characteristic axiom holds in this model. This model's biggest advantage is the matrix representation of its transformations (see Chapter 6). However, both distances and angle measures are complicated in this model.

The Poincaré Model (1882). *Point* means a point in the interior of a particular Euclidean circle. *Line* means the portion interior to the given circle of any one of its diameters or of a Euclidean circle orthogonal to this circle. The chief advantage of this model is that angles are measured as they would be for curves in Euclidean geometry. Hence, as Fig. 3.6 illustrates, the angle sum of a triangle is less than 180°. The picture at the beginning of this chapter is based on the Poincaré model. Because angle measures are Euclidean, the repeated objects in this design are recognizably the same, although somewhat distorted to our eyes. The hyperbolic size of each of these creatures is the same in this model, which gives a sense of how distances are measured with the same complicated formula as the Klein model. In Section 4.6 we consider the transformations for this model and their connections with complex numbers.

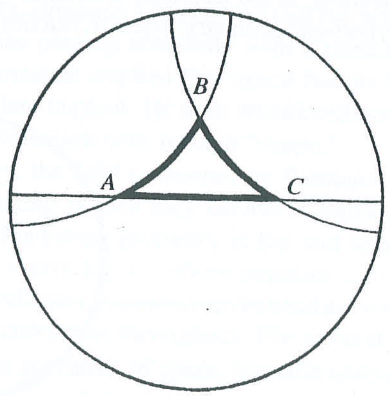

Figure 3.6 In the Poincaré model, the angle sum is less than 180°.

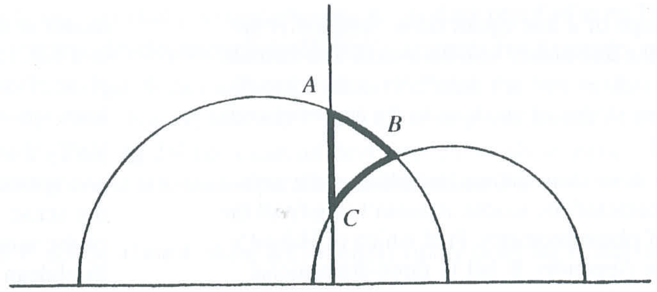

Figure 3.7 In the half-plane model, the angle sum is less than 180°.

The Half-Plane Model (1882). *Point* means a point in the upper half of the Euclidean plane [that is, points (x, y) with $y > 0$]. *Line* means the portion above the x-axis of either a vertical line or a circle with its center on the x-axis (Fig. 3.7). Poincaré developed this model and showed its close relation to the Poincaré model. This model also measures angles as Euclidean angles and, in addition, contains equations of lines that are easy to find. However, distances appear even more distorted than in the other models.

The Pseudosphere (1868). Eugenio Beltrami developed this first model of hyperbolic geometry. Figure 3.3 shows the surface of a pseudosphere. *Lines* are the paths of shortest distance along the surface (geodesics). Furthermore, both distances and angles are measured in a Euclidean manner. Thus this model is fairly natural. . The "equator" of the pseudosphere shown in Fig. 3.3 is an *ad hoc* gluing together of two portions (SEE BELOW.)

Unless explicitly stated, the figures in the rest of this chapter are not based on any of these models. This procedure will encourage thinking of this geometry as an entire system, not just an isolated exercise applying only to an artificial model. Many of the lines drawn in the figures will be curved so that they will not appear to intersect. This corresponds to one of the findings of Saccheri, mentioned earlier, that lines can approach one another in this geometry without intersecting.

PROBLEMS FOR SECTION 3.1

1. Use the Klein model of hyperbolic geometry to investigate how many lines through a point P do not intersect a (hyperbolic) line k that is not on P.

2. Use the points inside the (Euclidean) unit circle $x^2 + y^2 = 1$ for the Poincaré model.

a) Show that the circle $(x - 1.25)^2 + y^2 = (0.75)^2$ is orthogonal to the unit circle. Graph these two circles on the same axes.

b) Find the intersections (inside the unit circle) of the circle from part (a) with $y = (9/13)x$ and $y = -(9/13)x$, two diameters of the unit circle. Include these two lines in the graph for part (a) to form a triangle.

c) Verify visually that the angle sum of the triangle in part (b) is less than 180°.

d) Find the measures of the three angles of the triangle in part (b) and verify that their sum is less than 180°. [*Hint:* The angle made by a line and a circle is the angle made by that line and the tangent to the circle at their intersection.

"It is possible to extend either portion, but the surface is no longer 'analytic.' In particular, it becomes increasingly wrinkled."

We can think of two vertical lines as meeting at 'infinity,' outside of the model, and so they are also sensed parallels to each other.

The slope of a line equals $\tan\alpha$, where α is the angle the line makes with the x-axis. The formula $\tan(\alpha - \beta) = (\tan\alpha - \tan\beta)/(1 + \tan\alpha\tan\beta)$ converts slopes of two lines to the angle between them.]

3. Although three-dimensional Euclidean space satisfies the characteristic axiom, it doesn't satisfy all the axioms of plane geometry. Find which of Hilbert's axioms in Appendix B fail in three-dimensional Euclidean space.

4. Use the half-plane model of hyperbolic geometry.

a) Find the equation of the line that is on the points (1, 1) and (4, 2).

b) Explain why the equations of lines in this model are either $x = c$ or $y = \sqrt{r^2 - (x - c)^2}$, for appropriate constants c and r.

c) Use a Euclidean argument to explain why two points in this model have exactly one line which is on both of them.

d) Verify that the point $(-4, 10)$ is not on the line $y = \sqrt{25 - (x + 4)^2}$. Find the equations of two lines on $(-4, 10)$ that do not intersect the line $y = \sqrt{25 - (x + 4)^2}$. Graph these three lines.

e) Use a Euclidean argument to explain why the characteristic axiom always holds in this model. [*Hint:* Consider the two kinds of lines separately.]

f) Verify that the two lines $y = \sqrt{25 - x^2}$ and $y = \sqrt{16 - (x - 1)^2}$ do not have a point of the model in common, although they do intersect in a Euclidean point as curves in the Euclidean plane. Find this Euclidean point. We call these lines *sensed parallels.*

g) Verify that every point in the model has at least one sensed parallel to the line $y = \sqrt{25 - x^2}$ in the sense that the only Euclidean intersection of the sensed parallel with $y = \sqrt{25 - x^2}$ is the Euclidean point you found in part (e). Explain why there are exactly two sensed parallels to a line through a point not on that line.

5. (Calculus) In the half-plane model of hyperbolic geometry, consider the triangle formed by the lines $y = \sqrt{5 - (x - 3)^2}$, $y = \sqrt{10 - (x - 4)^2}$, and $x = 4$. Find the three vertices of this triangle and graph it. Find the measures of the three angles of this triangle. [*Hint:* See Problem 2(d). Use calculus and trigonometry. The angle sum is approximately 161.6°.]

6. A curious property of hyperbolic geometry is that any two lines that do not intersect and are not sensed parallels (described informally in Problem 5) have a common perpendicular. Use the half-plane model of hyperbolic geometry to illustrate this property with the two lines $y = \sqrt{1 - x^2}$ and $x = 2$. Graph these lines first. Then use relevant Euclidean concepts, including orthogonal circles, to find the hyperbolic line perpendicular to both of them.

3.2 Properties of Lines and Omega Triangles

We develop hyperbolic geometry following the path of Saccheri, Gauss, Lobachevsky, and Bolyai, but adding the foundation of Hilbert and others. We can thus accentuate the geometric intuition of earlier work while having the logical basis required to avoid the flaws discovered later.

Our undefined terms are *point, line, on, between,* and *congruent*. Our axioms are Hilbert's axioms for plane geometry (see Appendix B), with the exception of axiom IV-1; Playfair's axiom, which we replace with the characteristic axiom of hyperbolic geometry. (See Section 3.1.) We also avail ourselves of Euclid's first 28 propositions, or those he proved without using the parallel postulate. (See Appendix A.) These propositions give us many of the key properties about lines and triangles, which we need in our development of hyperbolic geometry. They are provable in Hilbert's axiomatic system without our having to use either axiom IV-1 or the characteristic axiom.

Like the developers of hyperbolic geometry, we prove many theorems in this chapter, including Theorem 3.2.1, by contradiction. Until 1868, when Beltrami built the first model of hyperbolic geometry, mathematicians wondered whether the characteris-