b) Find and graph the fourth-degree polynomial described in Theorem 2.4.1 that goes through the points $(0,0)$, $(\frac{\pi}{2},1)$, $(-\frac{\pi}{2},-1)$, $(\pi,0)$, and $(-\pi,0)$.

c) Find and graph the cubic polynomials described in Theorem 2.4.2 that connect the points $(-\pi,0)$, $(-\frac{\pi}{2},-1)$, $(0,0)$, $(\frac{\pi}{2},1)$ and $(\pi,0)$ and have the same derivatives as $\sin x$ at those points.

d) Graph $y=\sin x$ and compare the advantages and disadvantages of each of the preceding approximations for $y=\sin x$.

2.5 Higher Dimensional Analytic Geometry

The formal, algebraic language of analytic geometry and vectors has enabled mathematicians to model higher dimensions as easily as two or three dimensions. The first investigations of geometry beyond three dimensions by Arthur Cayley and others starting in 1843 seemed puzzling and even nonsensical to most people, including many mathematicians. However, the variety and importance of applications in mathematics, physics, economics, and other fields in the twentieth century have provided convincing evidence of the significance and naturalness of higher dimensions in geometry. We briefly consider the analytic geometry of $\mathbf{R}^3$ before discussing polytopes, the higher dimensional analog of the polyhedra we considered in Section 1.6.

Definition 2.5.1 By a *point* in n-dimensional Euclidean geometry, we mean a vector $\vec{v}=(v_1,v_2,\dots,v_n)$ in $\mathbf{R}^n$. By a *line* we mean a set of points $\{\alpha\vec{u}+\vec{v}:\alpha\in\mathbf{R}\}$, where $\vec{u}$ and $\vec{v}$ are fixed vectors and $\vec{u}\neq 0$. A point $\vec{w}$ is *on* the line above iff for some α, $\vec{w}=\alpha\vec{u}+\vec{v}$. Two lines $\alpha\vec{u}+\vec{v}$ and $\beta\vec{s}+\vec{t}$ are *parallel* iff $\vec{s}$ is a scalar multiple of $\vec{u}$. The *distance* between two points $\vec{u}$ and $\vec{v}$ is $d(\vec{u},\vec{v})=\|\overrightarrow{u-v}\|=\sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\cdots+(u_n-v_n)^2}$. A *plane* is the set of points $\{\alpha\vec{u}+\beta\vec{v}+\vec{w}:\alpha,\beta\in\mathbf{R}\}$, where $\vec{u}$ and $\vec{v}$ are independent vectors and $\vec{w}$ is any fixed vector.

Remarks Figures 2.34 and 2.35 illustrate lines and planes in $\mathbf{R}^3$. In the definition of a line, we shift the line through the origin (or one-dimensional subspace) $\{\alpha\vec{u}:\alpha\in\mathbf{R}\}$

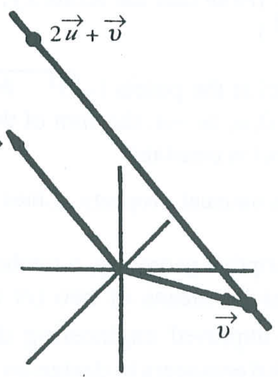

Figure 2.34 A line in $\mathbf{R}^3$.

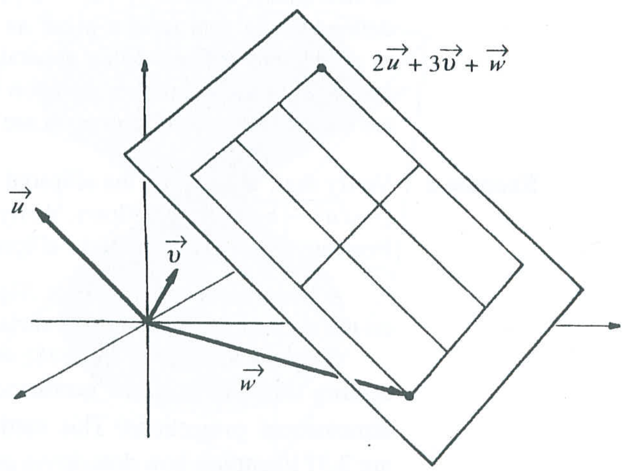

Figure 2.35 A plane in $\mathbf{R}^3$.

by adding the vector $\vec{v}$ so that we have a line through $\vec{v}$. Different vectors $\vec{v}$ give different parallel lines. The definition of a plane is similar to the definition of a line. The set $\{\alpha\vec{u}+\beta\vec{v} : \alpha, \beta \in \mathbf{R}\}$ is a two-dimensional subspace of $\mathbf{R}^n$. Adding the vector $\vec{w}$ translates this subspace to the parallel plane through $\vec{w}$. In $\mathbf{R}^3$, a plane can also be represented as the set of points (x, y, z) satisfying a linear equation $ax+by+cz+d=0$. In general, a linear equation in n variables, such as $a_1x_1+a_2x_2+\cdots+a_nx_n+d=0$, represents an $n-1$-dimensional *hyperplane* in $\mathbf{R}^n$. The formula for distance comes from the generalization of the Pythagorean theorem.

Exercise 1 Explain why the line through two points $\vec{s}$ and $\vec{t}$ consists of points of the form $\alpha(\overrightarrow{s-t})+\vec{t}$. [*Hint:* Consider $\alpha=0$ and $\alpha=1$.] Explain why the plane through three points $\vec{r}$, $\vec{s}$, and $\vec{t}$ consists of points of the form $\alpha(\overrightarrow{r-t})+\beta(\overrightarrow{s-t})+\vec{t}$.

Example 1 Use analytic geometry to verify that in three-dimensional geometry, if two distinct planes intersect, they intersect in a line.

Solution. Let's begin with two distinct planes $\alpha\vec{u}+\beta\vec{v}+\vec{w}$ and $\gamma\vec{r}+\delta\vec{s}+\vec{t}$. Then the coordinates of the six vectors are known, but the values of the four scalars can vary. If the point $\vec{a}=(a_1, a_2, a_3)$ is on both planes, the following system of three equations in the four unknowns α, β, γ, and δ has a solution.

$$\alpha u_1+\beta v_1+w_1=\gamma r_1+\delta s_1+t_1(=a_1).$$

$$\alpha u_2+\beta v_2+w_2=\gamma r_2+\delta s_2+t_2(=a_2).$$

$$\alpha u_3+\beta v_3+w_3=\gamma r_3+\delta s_3+t_3(=a_3).$$

However, as there are more unknowns than equations, elementary linear algebra assures us of infinitely many solutions. The planes are distinct, so these solutions must correspond to the points on a line, not a whole plane. •

Conics (Section 2.2) generalize to conic surfaces in three dimensions. Some of these surfaces, such as the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/b^2)=1$, the hyperboloid of two sheets $x^2/a^2-y^2/b^2-z^2/b^2=1$, and the paraboloid $z=ax^2+ay^2$, can be defined using foci (and a plane as a directrix for the paraboloid) in the same way as conics are defined. More generally, we define a conic surface as the set of points satisfying a second-degree equation in x, y, and z. (Note that the terms xy, xz, and yz are considered second degree, as are x^2, y^2, and z^2.)

Exercise 2 Verify that, if $a^2>b^2$, the ellipsoid must have foci at the points $(\sqrt{a^2-b^2}, 0, 0)$ and $(-\sqrt{a^2-b^2}, 0, 0)$ as follows. Verify that, for $x=0$, a, or $-a$, the sum of the distances from any point (x, y, z) on the ellipsoid to these foci is constant.

A hyperboloid of one sheet (Fig. 2.36) has an unusual property: Lines lie entirely on the surface, even though the surface is curved.

Gaspard Monge (1746–1818) developed descriptive geometry, a method of representing three-dimensional geometric constructions by means of two (or more) two-dimensional projections. This method radically improved engineering design. Figure 2.37 illustrates how descriptive geometry enabled engineers to design on paper exact plans for their constructions. Computer-aided design now supersedes this hand-drawn approach but is based on the same analytic geometry.

GASPARD MONGE

At age 22, Gaspard Monge (1746–1818) became a professor of mathematics and at 25, a professor of physics. His brilliance and gift as a teacher were recognized throughout his life. He supported the French Revolution and made important contributions to it and to Napoleon's regime that followed. About the time the French Revolution erupted (1789), Monge developed descriptive geometry, leading to accurate two-dimensional engineering drawings of three-dimensional figures. The new government recognized the military importance of this method and classified it top secret. Monge served in various government positions, including a ministerial post. His greatest contribution was the founding in 1795 and the sustaining of the Ecole Polytechnique. Monge contributed greatly to our understanding of education with the organization of this renowned school and his own teaching. Monge and Napoleon became friends and, upon Napoleon's exile, Monge lost his official positions. No official notice was given of his death soon after, but his students mourned him.

In addition to his development of descriptive geometry, Monge was instrumental in championing geometric methods in mathematics following a period of dominance by algebra and calculus. He recognized the need for both analytic and synthetic geometric approaches and revived interest in projective geometry. He also made major contributions to what we now call differential geometry, making it into a separate branch of mathematics.

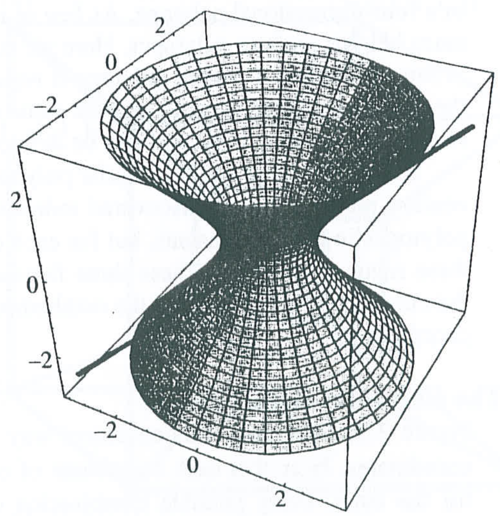

Figure 2.36 A hyperboloid of one sheet.

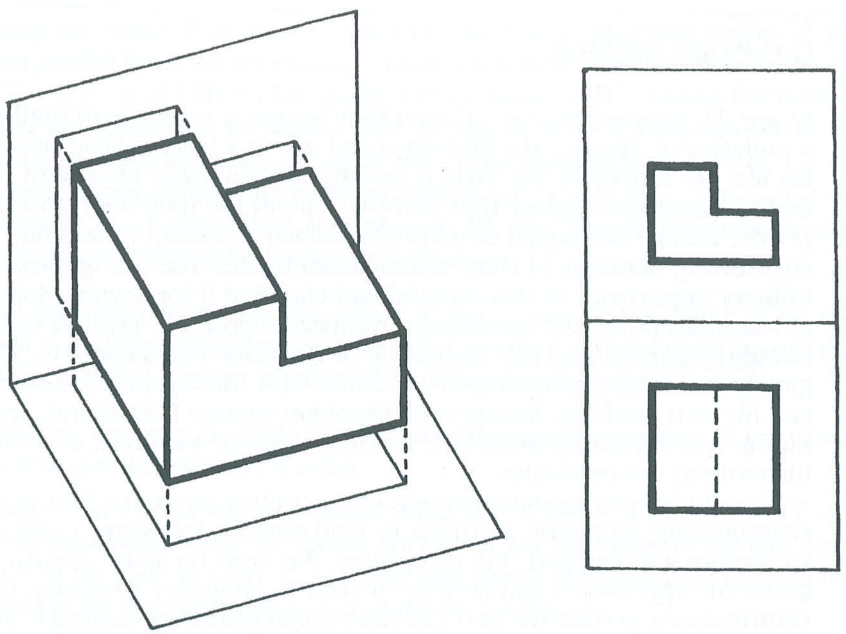

Figure 2.37 Two projections of a simple shape.

2.5.1 Regular polytopes

Mathematicians have studied polygons in two dimensions and polyhedra in three dimensions for millennia. By 1850 some of the earliest geometry of higher dimensions studied the analog of these shapes, called *polytopes*. A polyhedron is constructed by attaching polygons at their edges. Similarly, attaching polyhedra at their faces results in a four-dimensional polytope. As few of us can imagine such a construction, coordinates help us analyze polytopes. Here we consider only some of the regular polytopes. A four-dimensional convex polytope is *regular* provided that all the polyhedra in it are the same regular polyhedron and the same number of polyhedra meet at every vertex. Higher dimensional polytopes can be defined similarly. (See Coxeter [3, Chapter 22].)

Recall that there are five regular polyhedra in three dimensions. Several nineteenth century mathematicians discovered independently that there are a total of six regular polytopes in four dimensions, but for each dimension greater than four, there are only three regular polytopes. These three families of regular polytopes are the analogs of the cube, the tetrahedron, and the octahedron, which fortunately have simple Cartesian coordinates.

The Cube and Hypercubes.

Figure 2.38 illustrates one convenient way to place a square and a cube in Cartesian coordinates. Note that each coordinate of each point is either 1 or -1. Furthermore, for the cube, every possible combination of three such coordinates appears among the $8 = 2^3$ points. This suggests a way to generalize to four dimensions. Consider the $16 = 2^4$ points in $\mathbf{R}^4$ whose coordinates are all ± 1, some of which are labeled in

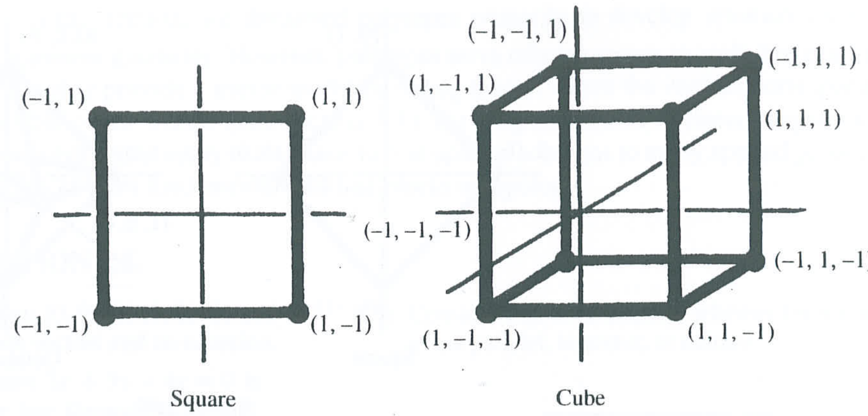

Figure 2.38

Fig. 2.39. These points are the vertices of a *hypercube*. The distance between two points joined by an edge in Fig. 2.39 is 2. Each point has four neighbors at this distance. The other possible distances between points on the hypercube are $2\sqrt{2}$, $2\sqrt{3}$, and 4.

Exercise 3 Find points on the hypercube at distances 2, $2\sqrt{2}$, $2\sqrt{3}$, and 4 from $(1, 1, 1, 1)$.

The Octahedron and Cross Polytopes.

The placement of the square and the regular octahedron in Fig. 2.40 suggests a way to generalize these figures. In $\mathbf{R}^4$, consider the eight points with 0 for three of their coordinates and 1 or -1 for the other coordinate. We designate these eight points as the vertices of the regular four-dimensional analog, called a *cross polytope*. Connect two of

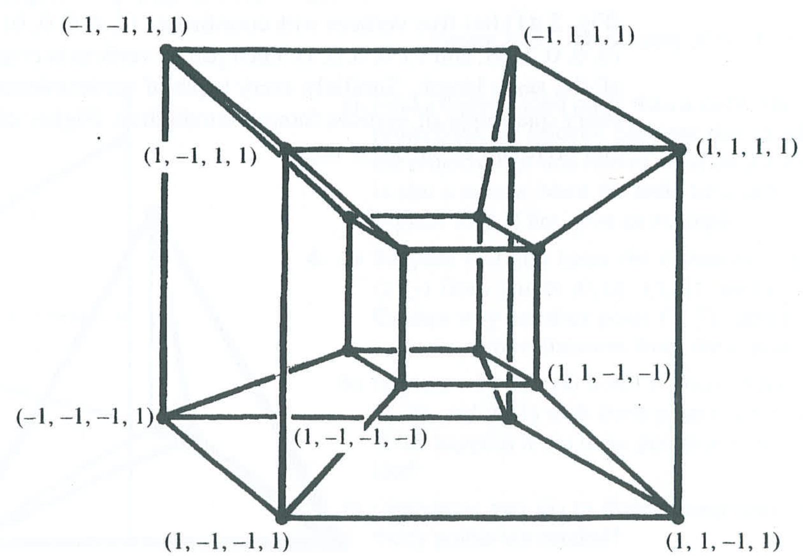

Figure 2.39 A hypercube.

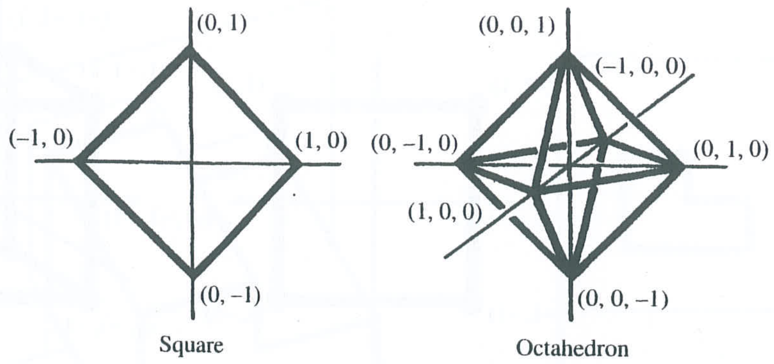

Figure 2.40

these vertices with an edge if their nonzero values appear in different coordinates. Thus $(0, 1, 0, 0)$ and $(0, 0, -1, 0)$ are connected with an edge, but $(0, 1, 0, 0)$ and $(0, -1, 0, 0)$ are not connected.

Exercise 4 Find the two possible distances between vertices for the four-dimensional analog of the octahedron. Verify that each vertex is on six edges.

The Tetrahedron and Regular Simplexes.

The rectangular coordinate system doesn't give easy three-dimensional coordinates for a regular tetrahedron. However, Fig. 2.40 suggests another approach. The octahedron has a triangular face with particularly simple coordinates: $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Similarly, in four dimensions $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$ are the vertices of a regular tetrahedron. Each vertex is a distance of $\sqrt{2}$ from the others. Now we can describe the regular *four-dimensional simplex*, as it is known. This shape (Fig. 2.41) has five vertices with coordinates $(1, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0)$, $(0, 0, 1, 0, 0)$, $(0, 0, 0, 1, 0)$, and $(0, 0, 0, 0, 1)$. Each pair of vertices is connected for a total of 10 edges, all the same length. Similarly, every triple of vertices forms an equilateral triangle, and every quadruple of vertices forms a tetrahedron. Higher dimensional regular simplexes can be found in the same way.

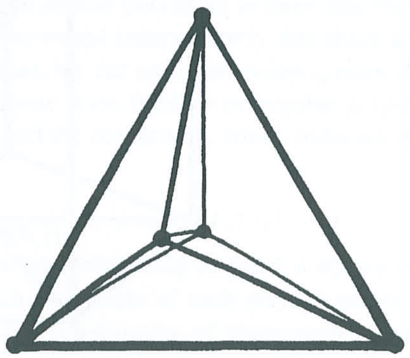

Figure 2.41 A regular four-dimensional simplex.

In this section, we discussed polytopes primarily to develop intuition for four-dimensional geometry. However, polytopes serve other purposes as well. For example, hypercubes provide a useful model in coding theory, where the vertices correspond to possible coded words. (See Section 7.3.) The simplex method in linear programming uses polytopes in many dimensions to find optimal solutions to many applied problems. Simplices form fundamental building blocks in topology.

PROBLEMS FOR SECTION 2.5.

1. a) Find the plane through $(0, 0, 0)$, $(1, 2, 4)$, and $(2, 0, 0)$ by using both vectors and an equation.

b) Explain why the plane $2x + 3y - 4z = 0$ is perpendicular to the line through $(2, 3, -4)$ and the origin in three dimensions. [*Hint:* Use vectors.] Is $2x + 3y - 4z + 5 = 0$ also perpendicular to that line? Explain.

2. If two planes in $\mathbf{R}^4$ intersect in at least one point, must they intersect in more than that one point? (See Example 1.) If so, show why. If not, find an example of two planes in $\mathbf{R}^4$ intersecting in only one point.

3. (Descriptive geometry) In this problem you are to project onto the xy-plane and the yz-plane; that is, a point (x, y, z) will project to (x, y) and (y, z), and a line k has two projections, k_1 and k_2 (Fig. 2.42). The yz-projection directly above the xy-projection indicates that the y-coordinates match.

a) Redraw Fig. 2.42 and include the line parallel to k through the point P. Explain why your line is parallel to k.

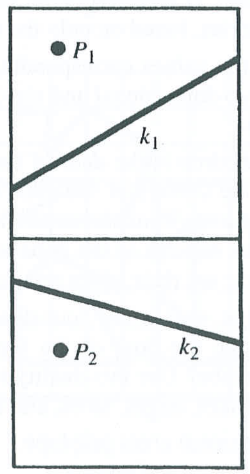

Figure 2.42 Projections of a point and a line.

b) Consider Fig. 2.43. Explain whether lines k and m are parallel, intersect, or neither.

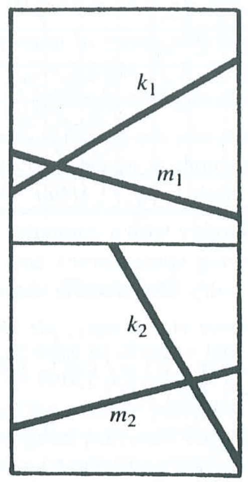

Figure 2.43 Projections of two lines.

c) Find a convex solid other than a cube whose two projections are squares. Suppose that in addition the projection of this convex solid on the xz-plane is also a square. Must the solid be a cube? If so, explain why. If not, give an example.

4. a) Suppose that you know the distances of point (x, y) from points $(0, 0)$, $(1, 0)$, and $(0, 1)$. Explain why no other point $(\overline{x}, \overline{y})$ can be at the same respective distances from these points.

b) Explain why part (a) holds if you replace $(0, 0)$, $(1, 0)$, and $(0, 1)$ with three points not on a line. What happens if the three points are on the same line?

c) Generalize part (a) to three dimensions. How many points are needed?

d) Redo part (c) in n dimensions.

5. a) Define an ellipsoid, a hyperboloid of two sheets, and a paraboloid, based on the corresponding definitions from Definition 2.2.1.
   b) Derive the text's equations for the conic surfaces of part (a).
6. a) Explain why the equation of the unit sphere is $x^2 + y^2 + z^2 = 1$.
   b) Explain why the equation of a great circle on the unit sphere is $ax + by + cz = 0$, where not all a, b, and c are zero. (See Section 1.6.)

   Represent a great circle by its triple $[a, b, c]$ and a point by (x, y, z). Then (x, y, z) is on $[a, b, c]$ provided that $ax + by + cz = 0$.

   c) Find the two points of intersection of the great circles $[2, 2, 1]$ and $[2, -2, 1]$. Find the great circle through the points $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ and $(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})$.
   d) Explain why the point (p, q, r) is perpendicular (orthogonal) as a vector to every point on the great circle $[p, q, r]$. [*Hint:* Use linear algebra.]
7. Plot, preferably with a computer graphics package, the following space curves and surfaces given parametrically. Describe the shapes in words.
   a) The curve $x(t) = \cos t$, $y(t) = \sin t$ and $z(t) = t$.
   b) The curve $x(t) = t \cos t$, $y(t) = t \sin t$ and $z(t) = t$, for $t \geq 0$.
   c) The surface $x(u, v) = \sqrt{1 - u^2} \cos v$, $y(u, v) = \sqrt{1 - u^2} \sin v$ and $z(u, v) = u$, for $-1 \leq u \leq 1$.
   d) The surface $x(u, v) = \sqrt{1 + u^2} \cos v$, $y(u, v) = \sqrt{1 + u^2} \sin v$ and $z(u, v) = u$.
8. The hyperboloid $x^2 + y^2 - z^2 = 1$ (See Fig. 2.36) has lines that lie on its surface. Start with a line through $(1, 0, 0)$ and a point on the hyperboloid at a height of $z = \sqrt{3}$.
   a) Explain why points on the hyperboloid with $z = \sqrt{3}$ can be written as $(2 \cos \theta, 2 \sin \theta, \sqrt{3})$. Explain why the line through this point and $(1, 0, 0)$ is given by $\alpha(2 \cos \theta - 1, 2 \sin \theta, \sqrt{3}) + (1, 0, 0)$.
   b) Explain why, if the line in part (a) is on the surface of the hyperboloid, it is also on $(2 \cos -\theta, 2 \sin -\theta, -\sqrt{3}) = (2 \cos \theta, -2 \sin \theta, -\sqrt{3})$. Show that θ satisfies $\cos \theta = \frac{1}{2}$. Find the two values of $\sin \theta$ and determine the two lines.
   c) Prove that every point on the lines in part (b) lies on the hyperboloid.
   d) Explain why an appropriate rotation of the two lines you found will give two lines through every point on the hyperboloid.
   e) Make a model of this hyperboloid by stretching strings between two circles. When one circle is rotated over the other, the strings lean to form a hyperboloid (Fig. 2.44).

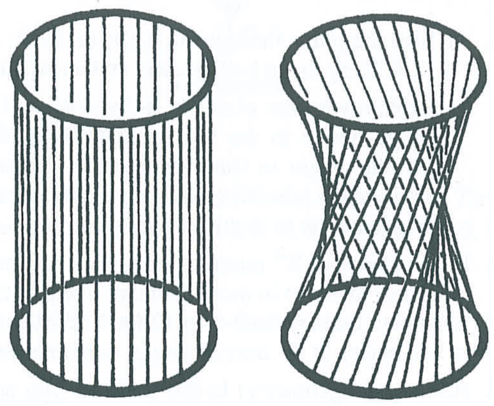

Figure 2.44

9. a) An edge, a square, a cube, and a hypercube are analogous shapes in one to four dimensions. For each, count the edges per vertex, faces per vertex, and so on, and the total number of vertices, edges, faces, and so on. Make a table of this information.
   b) Look for patterns, and explain any patterns you find. If possible, write formulas for these categories, based on only the dimension.
   c) Find the values corresponding to part (a) for the five-dimensional and six-dimensional hypercubes.
10. The octahedron is the *dual* of the cube: when you connect the centers of the cube's faces, you get an octahedron. Thus the octahedron has the same number of vertices as the cube has faces, and vice versa. How are their edges related?
   a) In what way is the four-dimensional cross polytope the dual of the four-dimensional hypercube? Use this duality to find the number of vertices, edges, faces, and octahedra of a four-dimensional cross polytope.
   b) Repeat part (a) for the five-dimensional cross polytope.

PROJECTS FOR CHAPTER 2

1. Define and investigate taxicab conics and other topics in taxicab geometry. (See Krause [8].) Note: Different positions of the foci and different slopes of the directrix lead to different-looking ellipses, hyperbolas, and parabolas.
2. Pick's theorem gives the area of many *lattice polygons*, that is, polygons whose vertices have integer coordinates, called *lattice points*.
   a) For each lattice polygon shown in Fig. 2.45, find its area and count the number B of lattice points on the boundary of the polygon and the number I of lattice points in the interior of the polygon. Find an equation relating the area with the numbers B and I. Does this equation hold for the polygons shown in Fig. 2.46?

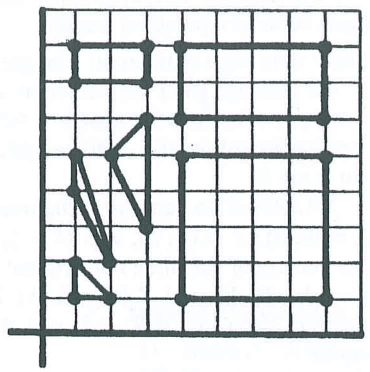

Figure 2.45

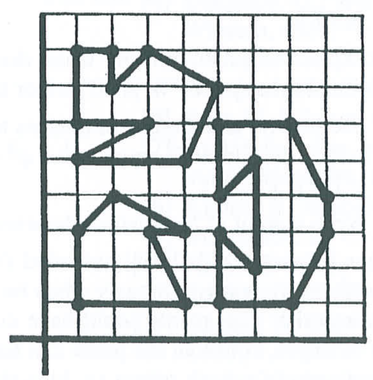

Figure 2.46

   b) Prove your equation for lattice rectangles parallel to the axes.
   c) Do the lattice shapes shown in Fig. 2.47 satisfy your equation? Explain how these polygons differ from those in Figs. 2.45 and 2.46. State Pick's theorem, incorporating any needed hypotheses.

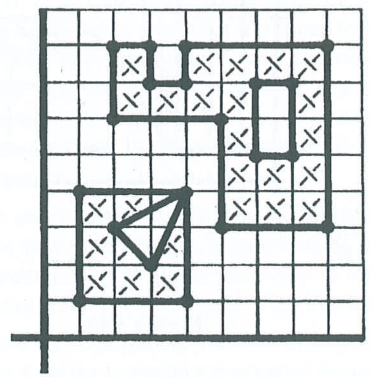

Figure 2.47

   d) Add an interior edge to divide a lattice polygon into two smaller lattice polygons (Fig. 2.48). How do B and I for the smaller polygons compare to those of the original polygon? How far can you carry this process of dividing? What can you say about the smallest lattice polygons? (See Coxeter [3, 209] for a proof of Pick's theorem.)

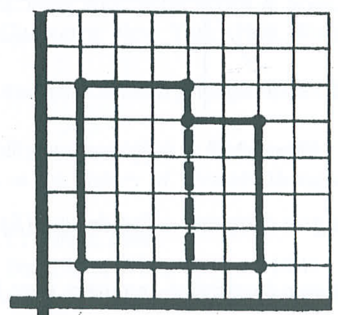

Figure 2.48

   e) Develop, state, and prove a restriction of Pick's theorem in three dimensions to rectangular boxes parallel to the axes (Fig. 2.49). Does this restriction work for pyramids? (Fig. 2.50.) Reeve [10] proves a general three-dimensional version.

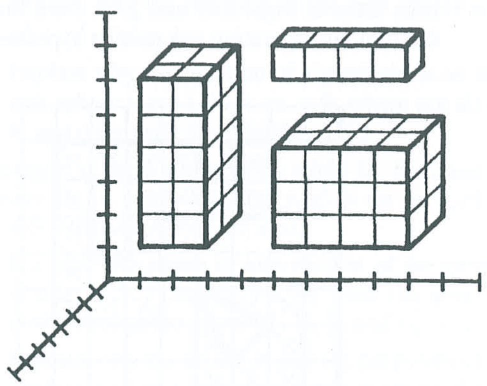

Figure 2.49

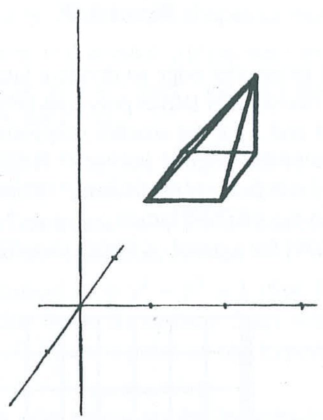

Figure 2.50 A pyramid with height 1.

3. Investigate computer-aided design. (See Mortenson [9].)

4. Investigate descriptive geometry. (See Douglass and Hoag [4].)

5. Investigate polytopes and four-dimensional geometry. (See Coxeter [3, Chapter 22] and Rucker [11].)

6. (Calculus) The folium of Descartes ($x^3 - 2xy + y^3 = 0$, shown in Fig. 2.1) is a challenge to graph by hand, even with the aid of calculus.

a) Use implicit differentiation to find dy/dx.

b) Find the values of x and y for which $dy/dx = 0$.

c) When is dy/dx undefined? At these points the curve has a vertical tangent. How are these points related to those in part (b)?

d) Find the points on the curve where $x = y$. Why is the curve symmetric with respect to this line?

e) Graph this curve, using the previous information and Newton's method to plot points on the curve.

f) Graph the folium of Descartes parametrically, using the following tricky substitution. In $x^3 - 2xy + y^3 = 0$, replace y by tx and solve for x to find $x(t)$. Then find $y(t) = tx(t)$. Verify that this graph matches the results in parts (a)–(e).

7. Napoleon's theorem. On each side of a triangle, construct an equilateral triangle lying outside (or inside) the triangle. The centers of these three triangles form an equilateral triangle.

a) Show that in an equilateral triangle with side x, the distance from the center to any vertex is $x/\sqrt{3}$. Also, rewrite $\cos(A + 60°)$ by using the formula $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

WLOG let the vertices of the triangle be $S = (0, 0)$, $T = (p, 0)$, and $U = (q, r)$. Call the centers of the three constructed equilateral triangles V, W, and X (Fig. 2.51). Napoleon's theorem states that VW, VX, and WX are equal.

b) Explain why showing that $VW = WX$ is sufficient.

c) Find ST, SU, and TU and then SV, SW, TW, and TX.

d) Explain why $\angle VSW$ is 60° wider than $\angle UST$ and similarly why $\angle XTW$ is 60° wider than $\angle UTS$.

e) Use part (a) and the law of cosines to verify that both VW^2 and WX^2 equal $(p^2 + q^2 + r^2 - pq + \sqrt{3}pr)/3$.

f) Write a proof of Napoleon's theorem.

8. Julius Plücker (1801–1868) extended the idea of a coordinate system enormously when he realized that entities other than points could have coordinates. For example, circles in the plane can have "circular coordinates" (a, b, r), where (a, b) is the center and $r > 0$ is the radius of the circle. Thus the set of circles in the plane is, in a sense, three dimensional.

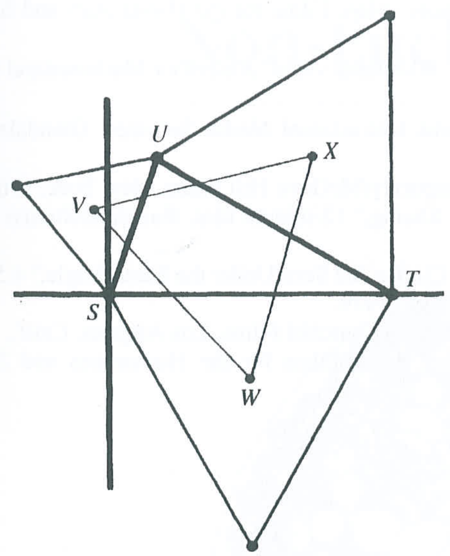

Figure 2.51 Napoleon's theorem.

a) Find the circular coordinates for the circle $x^2 + y^2 + 4x - 6y - 3 = 0$.

b) Find the equation of a circle having circular coordinates $(5, -4, 7)$.

c) Devise a set of "linear coordinates." How many "dimensions" does the set of lines in the plane have? Find, in your linear coordinates, the coordinates of the line whose equation is $y = 3x + 5$. Can two distinct pairs of your linear coordinates represent the same line? Are there any lines that do not have coordinates in your system? (Consider $x = 4$.) Are there any subsets of coordinates in your system that do not represent lines? (See Eves [6] for more information.)

9. Write an essay exploring the meaning of geometry in four or more dimensions. Suggestions: Compare multidimensional with plane geometry in a world that is strictly three dimensional. (See Rucker [11].)

10. Write an essay comparing analytic and synthetic geometry as ways to explain geometric concepts.

11. Write an essay comparing the certainty of algebraic derivations of geometric properties with the certainty of proving those properties in an axiomatic system. (See Grabiner [7].)

Suggested Readings

[1] Boyer, C. *History of Analytic Geometry*, New York: Scripta Mathematica, 1956.

[2] Broman, A., and L. Broman. Museum exhibits for the conics. *Mathematics Magazine*, 1994, 67(3):206–209.

[3] Coxeter, H. *Introduction to Geometry*, 2d ed. New York: John Wiley & Sons, 1969.

[4] Douglass, C., and A. Hoag. *Descriptive Geometry*. New York: Holt, Rinehart and Winston, 1962.

[5] Edwards, C., and D. Penney. *Calculus with Analytic Geometry*, Englewood Cliffs, N.J.: Prentice Hall, 1994.

[6] Eves, H. *A Survey of Geometry*, vols. I and II. Boston: Allyn & Bacon, 1965.

[7] Grabiner, J. The centrality of mathematics in the history of Western thought. *Mathematics Magazine*, 1988, 61(4):220–230.

[8] Krause, E. *Taxicab Geometry*. Mineola, N.Y.: Dover, 1986.

[9] Mortenson, M. *Geometric Modeling*. New York: John Wiley & Sons, 1985.

[10] Reeve, J. On the volume of lattice polyhedra. *Proceedings of the London Mathematical Society*, 1957, 3(7):378–395.

[11] Rucker, R. *Geometry, Relativity and the Fourth Dimension*. Mineola, N.Y.: Dover, 1977.

[12] Strang, G. *Linear Algebra and its Applications*, 2d ed. New York: Academic Press, 1980.

Suggested Media

1. "Applications of Conic Sections," 10-minute video, Films for the Humanities and Sciences, Princeton, N.J., 1996.
2. "Curves from Parameters," 24-minute video, Films for the Humanities and Sciences, Princeton, N.J., 1996.
3. "Descartes and Problem Solving," 60-minute video, American Mathematical Society, Providence, R.I., 1992.
4. "Dimension," 13-minute film, Aims Instructional Media Services, Glendale, Calif., 1970.
5. "Flatland," 12-minute film, Contemporary/McGraw Hill Films, New York, 1965.
6. "The Hypercube: Projections and Slicing," 12-minute film, Banchoff/Strauss Productions, Providence, R.I., 1978.
7. "Locus of Points from Which Two Circles Are Seen Under the Same Angle," 4.5-minute film, Educational Solutions, New York, 1979.
8. "Mathematical Curves," 10-minute film, Churchill Films, Los Angeles, Calif., 1977.
9. "Reflecting on Conics," 24-minute video, Films for the Humanities and Sciences, Princeton, N.J., 1996.