a) Describe all coordinates for the north and south poles.

b) Describe the curves on the sphere for $y = a$ and $x = b$.

c) Describe the curves on the sphere for $y = mx + b$.

d) If you interpret *point* as a point on the sphere and *line* as any of the curves of parts (b) and (c), which of Hilbert's axioms from group I (Appendix B) are true? Explain.

11. Assign coordinates to the points on the surface of a *torus* (a doughnut) using longitude and latitude measured in degrees (Fig. 2.23). Every point (x, y) has multiple representations, such as $(x + 360°, y + 720°)$. Interpret *point* as a point on the surface of a torus and *line* as the set of points satisfying an equation of the form $ax + by + c = 0$.

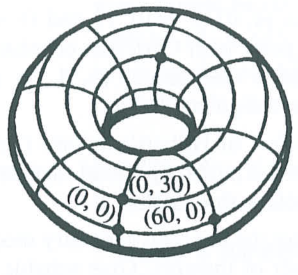

Figure 2.23 Coordinates on a torus.

a) Find values of a, b, and c so that the line $ax + by + c = 0$ is finitely long. Find other values so that that line is infinitely long.

b) Find a line that intersects n times with $y = 0$. Find a line that intersects $y = 0$ infinitely many times.

c) Which of Hilbert's axioms in group I and IV (Appendix B) hold in this model? Explain.

12. Taxicab geometry is an alternative analytic model with a distance function corresponding to the distances taxicabs travel on a rectangular grid of streets. The *taxicab distance* between $A = (a, b)$ and $S = (s, t)$ is $d_T(A, S) = |a - s| + |b - t|$. This geometry has quite different properties from Euclidean geometry. (See Section 1.4.)

a) How are two points that have the same Euclidean and taxicab distances related? If these distances are different, which is larger? If the smaller distance is 1, how large can the other distance be? Explain and illustrate.

b) A *taxicab circle* is the set of points at a fixed taxicab distance from a point. Describe taxicab circles. What are the possible types of intersection of two taxicab circles? Illustrate each.

c) A *taxicab midpoint* M of A and B satisfies $d_T(A, M) = d_T(M, B) = \frac{1}{2}d_T(A, B)$. Illustrate the different sets of midpoints two points can have.

d) The *taxicab perpendicular bisector* of two points comprises of all points equidistant from the two points. Describe and illustrate the different types of taxicab perpendicular bisectors.

e) In the plane find four points whose taxicab distances from each other are all the same.

13. Another distance formula on $\mathbf{R}^2$ uses the maximum of the x and y distances: For $A = (a, b)$ and $S = (s, t)$, $d_M(A, S) = Max\{|a - s|, |b - t|\}$. Redo Problem 12, using d_M.

2.4 Curves in Computer-Aided Design

Computer graphics depend heavily on analytic geometry. The computer stores the coordinates of the various points and information about how they are connected. The easiest connections are line segments and arcs of circles, which are easily described with elementary analytic geometry. However, connecting a series of points smoothly requires calculus and more advanced techniques, which we discuss briefly in this section. Computer-aided design (CAD) uses polynomials—a flexible and easily computed family of curves. For a sequence of points, we need a curve or a sequence of curves that smoothly passes through the points in the specified order. A simplistic approach finds one polynomial through all the points, as in the first two examples.

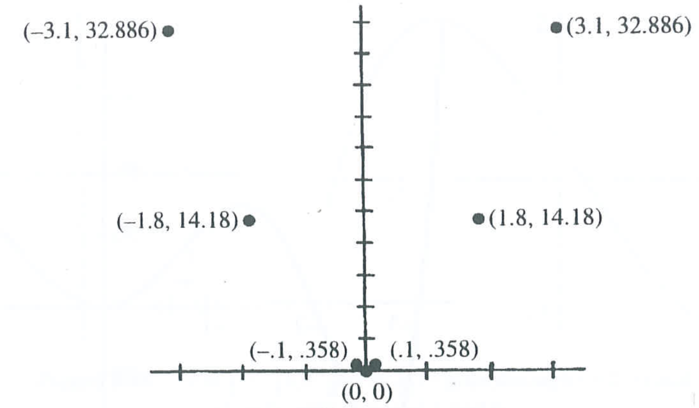

Figure 2.24 The given points for Examples 1, 2, and 3.

Example 1 Find the parabola $y = ax^2 + bx + c$ that goes through the three points $(0, 2)$, $(1, 3)$, and $(3, -7)$.

Solution. When we replace x and y by the values for each point we get three first-degree equations in a, b, and c: $2 = c$, $3 = a + b + c$, and $-7 = 9a + 3b + c$. Solving this system of three linear equations in three unknowns gives $y = -2x^2 + 3x + 2$. ●

Theorem 2.4.1 Given $n + 1$ points $P_j = (x_j, y_j)$, for $j = 0$ to n, with no two x_j equal, there is exactly one nth degree polynomial $y = a_0 + a_1x + \cdots + a_nx^n$ such that for each j, $y_j = a_0 + a_1x_j + \cdots + a_nx_j^n$.

Proof. See Strang [12, 80]. ■

Example 2 For the seven points shown in Fig. 2.24, Fig. 2.25 shows a freehand curve connecting these points with no unneeded bumps. Figure 2.26 shows that the graph of the

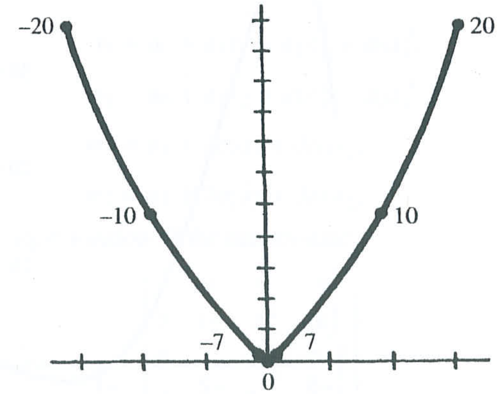

Figure 2.25 A freehand curve through the points in Fig. 2.24, with slopes as indicated.

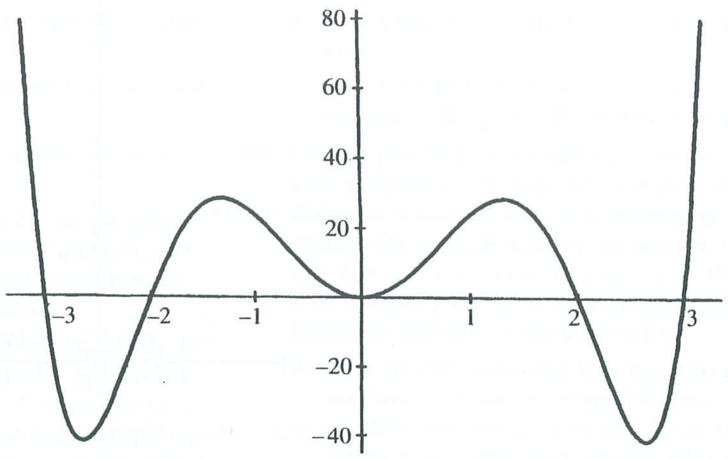

Figure 2.26 The curve $y = 35.93x^2 - 12.97x^4 + 0.998x^6$.

polynomial given by Theorem 2.4.1, $y = 35.93x^2 - 12.97x^4 + 0.9978x^6$, fits the free-hand curve poorly. ●

The method of Theorem 2.4.1 has serious drawbacks, starting with the poor fit of Example 2. A second drawback is that the addition of another point or the shift of one point necessitates completely recomputing the polynomial, as Fig. 2.27 illustrates. Third, with a large number of points, the computations become time-consuming and are prone to roundoff errors. Fourth, functions of y in terms of x cannot curve back on themselves or represent space curves. Hermite curves solve all these problems. Instead of one polynomial going through all the points, we find a sequence of $n - 1$ parametric

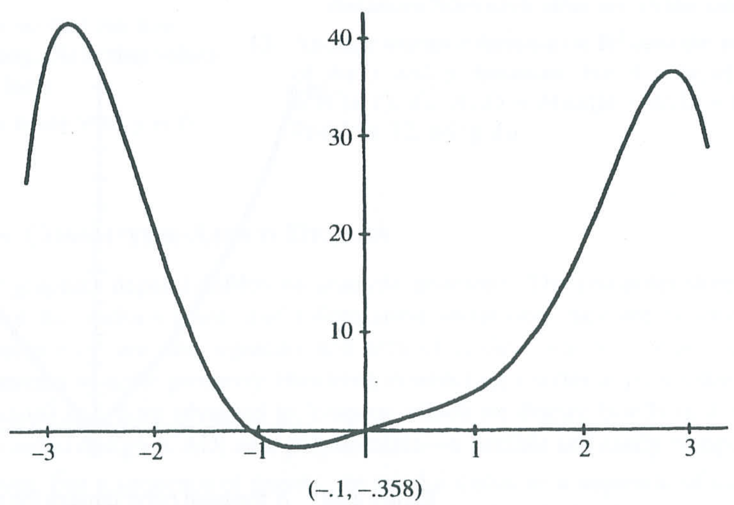

Figure 2.27 The curve resulting from shifting one point.

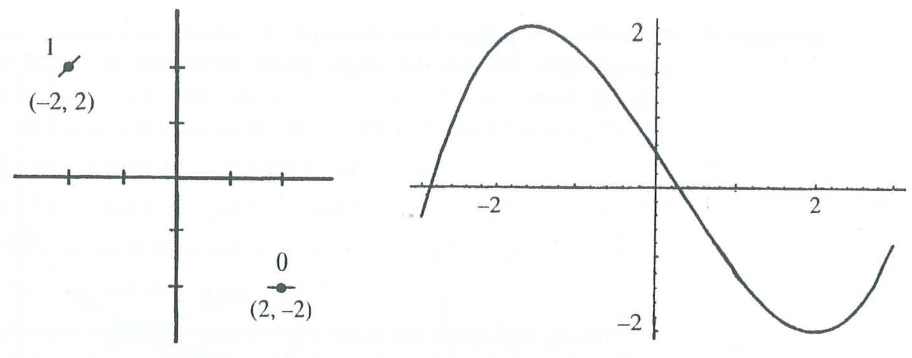

Figure 2.28 $y = \frac{1}{2} - \frac{7}{4}x - \frac{1}{8}x^2 + \frac{3}{16}x^3$ goes through $(-2, 2)$ and $(2, -2)$, with slopes of 1 and 0.

equations, using third-degree polynomials that connect neighboring points smoothly. That is, the derivative at the end of one curve is the derivative at the start of the next. The addition of a point in the middle or the repositioning of any point affects at most two cubics. Each cubic can be computed quickly, at least by a computer. Finally, the parametric form allows a curve to double back on itself, cross itself, or even twist into as many dimensions as are needed. Many applications use cubic spline curves, which build on the ideas presented here but do not need to be given the derivatives at the endpoints. (See Mortenson [9].) We motivate Hermite curves with the simpler, non-parametric situation of Theorem 2.4.2, illustrated in Fig. 2.28.

Theorem 2.4.2 Let two points be $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ with $x_1 \neq x_2$ and the desired slopes m_1 and m_2 at P and Q, respectively. Then there is a unique third-degree polynomial $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $f(x_1) = y_1$, $f(x_2) = y_2$, $f'(x_1) = m_1$, and $f'(x_2) = m_2$.

Proof. By elementary calculus, $f'(x) = a_1 + 2a_2x + 3a_3x^2$. The given points and slopes, P, Q, m_1, and m_2, determine four linear equations in the unknowns a_0, a_1, a_2, and a_3:

$$y_1 = a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3;$$

$$y_2 = a_0 + a_1x_2 + a_3x_2^2 + a_3x_2^3;$$

$$m_1 = a_1 + 2a_2x_1 + 3a_3x_1^2;$$

$$m_2 = a_1 + 2a_2x_2 + 3a_3x_2^2.$$

This system has a unique solution iff the determinant

$$\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{vmatrix}$$

does not equal zero. This determinant equals $-x_1^4 + 4x_1^3x_2 - 6x_1^2x_2^2 + 4x_1x_2^3 - x_2^4 = -(x_1 - x_2)^4$. Because $x_1 \neq x_2$, there is always a unique cubic. ■

Example 3 Consider the points from Example 2, which we connect with a sequence of cubic polynomials. We use the slopes given previously in Fig. 2.25. For example, for the interval from $x = -3.1$ to $x = -1.8$, we have $f(-3.1) = 32.886$, $f(-1.8) = 14.18$, $f'(-3.1) = -20$ and $f'(1.8) = -10$. Theorem 2.4.2 gives the system:

$$a_0 - 3.1a_1 + 9.61a_2 - 29.791a_3 = 32.886;$$
$$a_0 - 1.8a_1 + 3.24a_2 - 5.832a_3 = 14.18;$$
$$a_1 - 6.2a_2 + 28.83a_3 = -20;$$
$$a_1 - 3.6a_2 + 9.72a_3 = -10.$$

This system gives the cubic $y = -0.141 - 8.254x - 1.466x^2 - 0.723x^3$. The other cubics, from left to right, have for their equations $y = -0.340 - 6.96x + 0.154x^2 - 0.256x^3$, $y = 37.4x^2 + 16x^3$, $y = 37.4x^2 - 16x^3$, $y = -0.340 + 6.96x + 0.154x^2 + 0.256x^3$, and $y = -0.14 + 8.25x - 1.466x^2 + 0.723x^3$. Figure 2.29 shows how well these polynomials fit the path of the curve determined intuitively. ●

Hermite curves, defined parametrically, allow even greater flexibility than the curves of Theorem 2.4.2. The variables $x = x_1$ and $y = x_2$ (and x_3, etc., for more dimensions) are separately determined by cubic polynomials in t. As in Theorem 2.4.2, we find the cubic from the coordinates of the endpoints and the desired derivatives $x_i'(t)$ at each endpoint.

Theorem 2.4.3 Let $P_0 = (u_1, u_2, \ldots, u_n)$ and $P_1 = (v_1, v_2, \ldots, v_n)$ be any two points in $\mathbf{R}^n$ and assume that the derivatives $dx_i/dt = x_i'(t)$ are given at $t = 0$ and $t = 1$. Then for each x_i, there is a unique cubic function $f_i(t) = a_{0i} + a_{1i}t + a_{2i}t^2 + a_{3i}t^3$ such that $P_0 = (f_1(0), f_2(0), \ldots, f_n(0))$, $P_1 = (f_1(1), f_2(1), \ldots, f_n(1))$, $f_i'(0) = x_i'(0)$, and $f_i'(1) = x_i'(1)$.

Proof. Apply Theorem 2.4.2 with $t = x$ and $x_i = y$. ■

Definition 2.4.1 The parametric curve formed by the n-component functions $f_i(t)$ in Theorem 2.4.3 is a *Hermite curve*.

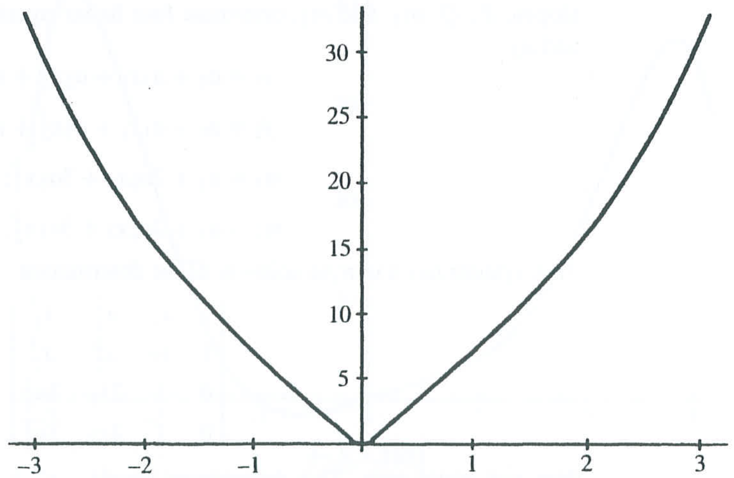

Figure 2.29 Six cubics fitting the original points.

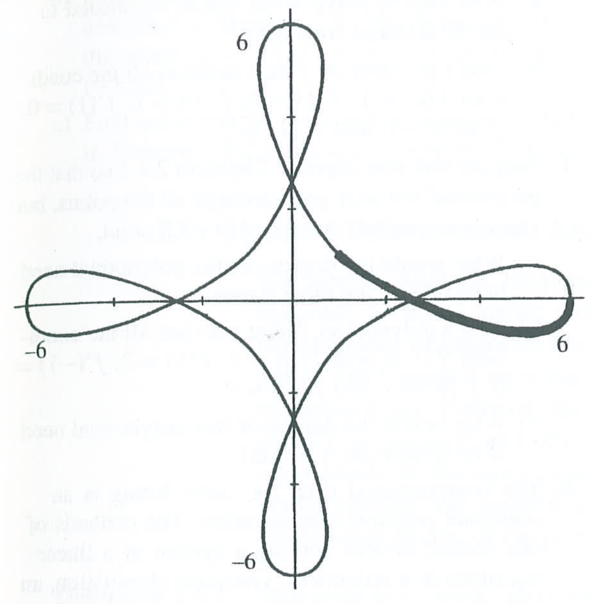

Figure 2.30

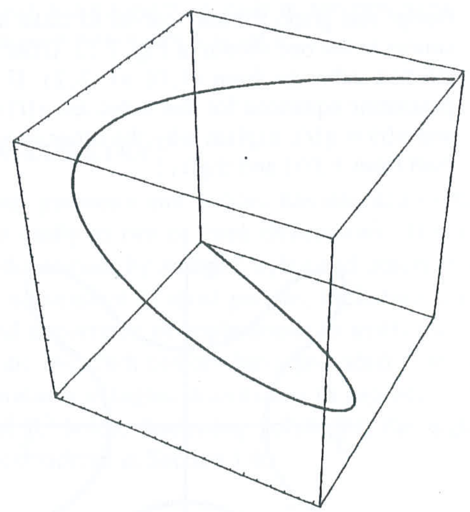

Figure 2.31 The graph of $x(t) = 16t^3 - 24t^2 + 9t$, $y(t) = 4t^2 - 4t$, and $z(t) = -2t^3 + 3t^2$.

Example 4 The shape shown in Fig. 2.30 contains eight Hermite curves. The thick portion of the curve starts at $(1, 1)$, with $x' = 3$ and $y' = -3$, and ends at $(6, 0)$, with $x' = 0$ and $y' = 1$. The parametric equations are $x(t) = 1 + 3t + 9t^2 - 7t^3$ and $y(t) = 1 - 3t - 2t^2 + 4t^3$. The parametric equations for the seven other portions of the curve shown in Fig. 2.30 use variations of these two functions. ●

Example 5 Figure 2.31 shows the graph of the three-dimensional Hermite curve $x(t) = 16t^3 - 24t^2 + 9t$, $y(t) = 4t^2 - 4t$, and $z(t) = -2t^3 + 3t^2$. ●

PROBLEMS FOR SECTION 2.4

Note: Computational devices are recommended for many of these problems.

1. Recompute the two cubics of Example 3 affected by the shift of the point $(-0.1, 0.358)$ to the new point $(-0.1, -0.358)$ with a slope of 1. Compare the graphs of these cubics with the one shown in Fig. 2.27.
2. Use Theorem 2.4.2 to find the following cubic polynomials, and then graph them.
   a) $f(-1) = -2$, $f(0) = 1$, $f'(-1) = 2$, and $f'(0) = 1$.
   b) $g(1) = -1$, $g(2) = 0$, $g'(1) = 0$, and $g'(2) = -1$.
   c) h defined so that it connects f and g smoothly.
3. Find and graph the following Hermite curves. Compare your result with that in Problem 2.
   a) $x(0) = -1$, $x(1) = 0$, $x'(0) = 1$, $x'(1) = 1$, $y(0) = -2$, $y(1) = 1$, $y'(0) = 2$, and $y'(1) = 1$.
   b) $x(0) = 1$, $x(1) = 2$, $x'(0) = 1$, $x'(1) = 1$, $y(0) = -1$, $y(1) = 0$, $y'(0) = 0$, and $y'(1) = -1$.
   c) $x(0) = 0$, $x(1) = 1$, $x'(0) = 1$, $x'(1) = 1$, $y(0) = 1$, $y(1) = -1$, $y'(0) = 1$, and $y'(1) = 0$.
   d) Give the equations for a new Hermite curve that is the mirror image in the x-axis of the one in part (a).

e) Give the equations for a new Hermite curve that is the result of rotating the one in part (a) 180° about the origin.

4. Design and graph Hermite curves to make a curve similar to the one shown in Fig. 2.32. [*Hint:* Make the first cubic go from $(1, 0)$ to $(2, 2)$. If the parametric equations for this cubic are $x(t) = f(t)$ and $y(t) = g(t)$, explain why the other curves are built from $\pm f(t)$ and $\pm g(t)$.]

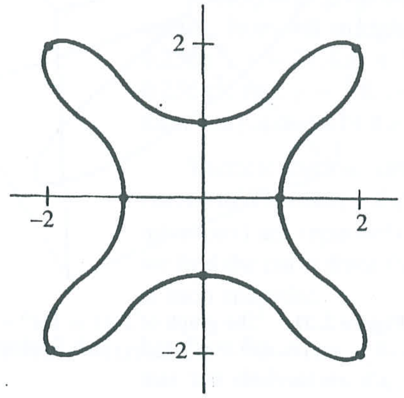

Figure 2.32

5. Repeat Problem 4 for the curve pictured in Fig. 2.33.

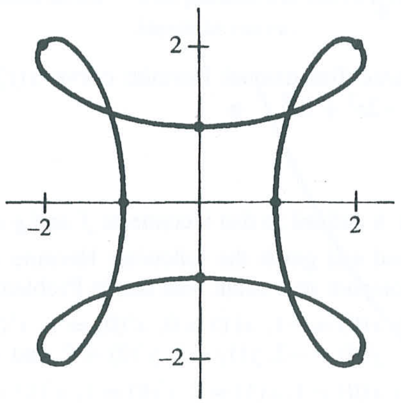

Figure 2.33

6. Suppose that you needed the curve in Theorem 2.4.2 to match given values for the second derivative at the start and endpoints, as well as the first derivative and heights there.

a) What degree polynomial would be needed to satisfy all these conditions?

b) Find a polynomial f that satisfies all the conditions $f(0) = 1$, $f(1) = 0$, $f'(0) = 1$, $f'(1) = 0$, $f''(0) = -1$, and $f''(1) = 2$.

7. Suppose that you improve Theorem 2.4.1 so that the polynomial not only goes through all the points, but also has a specified derivative at each point.

a) What would the degree of this polynomial need to be if there are three points?

b) Find a polynomial f that satisfies all the conditions $f(-1) = 0$, $f(0) = 1$, $f(1) = 2$, $f'(-1) = 0$, $f'(0) = 1$, and $f'(1) = 0$.

c) What would the degree of this polynomial need to be if there are n points?

8. The computational time for curve fitting is an important practical consideration. The methods of this section involve solving a system of n linear equations in n unknowns. Gaussian elimination, an efficient and widely used way to solve such systems, in general requires $(n^3 + 3n^2 - n)/3$ multiplications, which gives a good estimate of the computational time.

a) Use Theorem 2.4.1 to find the number of multiplications needed to identify one polynomial that goes through k points.

b) Repeat part (a) for the $k - 1$ cubics with designated slopes at these points, using Theorem 2.4.2.

c) Repeat part (b) for $k - 1$ two-dimensional Hermite curves.

d) Repeat part (c) for three-dimensional Hermite curves.

e) Explain why you need a polynomial of degree $2k - 1$ to find a curve that goes through k points and has a specified slope at each point. Repeat part (a) for this polynomial.

f) Compare the formulas you found in parts (a)–(e) and the ability of these curves to fit given conditions. How quickly does each formula grow as k increases? (Actual designs have dozens or hundreds of points.)

9. Compare Taylor polynomials and the polynomials of Theorems 2.4.1 and 2.4.2 as approximations for $y = \sin x$, as follows.

a) Find and graph from $-\pi$ to π the seventh-degree Taylor polynomial for $\sin x$ centered at 0.

b) Find and graph the fourth-degree polynomial described in Theorem 2.4.1 that goes through the points $(0, 0)$, $(\frac{\pi}{2}, 1)$, $(-\frac{\pi}{2}, -1)$, $(\pi, 0)$, and $(-\pi, 0)$.

c) Find and graph the cubic polynomials described in Theorem 2.4.2 that connect the points $(-\pi, 0)$, $(-\frac{\pi}{2}, -1)$, $(0, 0)$, $(\frac{\pi}{2}, 1)$ and $(\pi, 0)$ and have the same derivatives as $\sin x$ at those points.

d) Graph $y = \sin x$ and compare the advantages and disadvantages of each of the preceding approximations for $y = \sin x$.