7. For each of the following equations, identify the type of conic it is, determine whether it is degenerate, and sketch its graph.
   a) $\frac{1}{2}y^2 + x - 2y = 0$.
   b) $\frac{1}{4}x^2 + y^2 - x - 6y + 9 = 0$.
   c) $\frac{1}{4}x^2 + y^2 - x - 6y + 10 = 0$.
   d) $2x^2 - 3xy + y^2 - 4 = 0$. [*Hint:* Factor $2x^2 - 3xy + y^2 = 0$ and then explain why the factors are the asymptotes.]
   e) $x^2 - xy + y^2 - 1 = 0$. [*Hint:* Pick x-values and find the y-values.]

2.3 Further Topics in Analytic Geometry

The great flexibility of algebra and vectors fosters many geometric and applied variations on the analytic geometry model. We present a few here.

2.3.1 Parametric equations

Leonhard Euler introduced parametric equations in his 1748 textbook on analytic geometry. Separate functions for the x- and y-coordinates allow graphs of complicated curves. These functions, $x(t)$ and $y(t)$, are in terms of a third variable, t, which we can think of as representing time. The point $(x(t), y(t))$ traces out a curve as t varies. As y does not depend on x, the curve can double back on itself or even cross itself. We can use more parametric equations to describe curves in three or more dimensions.

Example 1 (Calculus) For $x(t) = t^3 - t$, $y(t) = t^2 - t^4$, and $-1.1 < t < 1.1$, the point $(x(t), y(t))$ traces out the curve shown in Fig. 2.15. The function $y(t)$ controls the heights of points. Setting the derivative $y'(t) = 2t - 4t^3$ equal to 0 shows the curve to reach its maxima and local minimum when $t = \pm\sqrt{2}/2$ and $t = 0$. ●

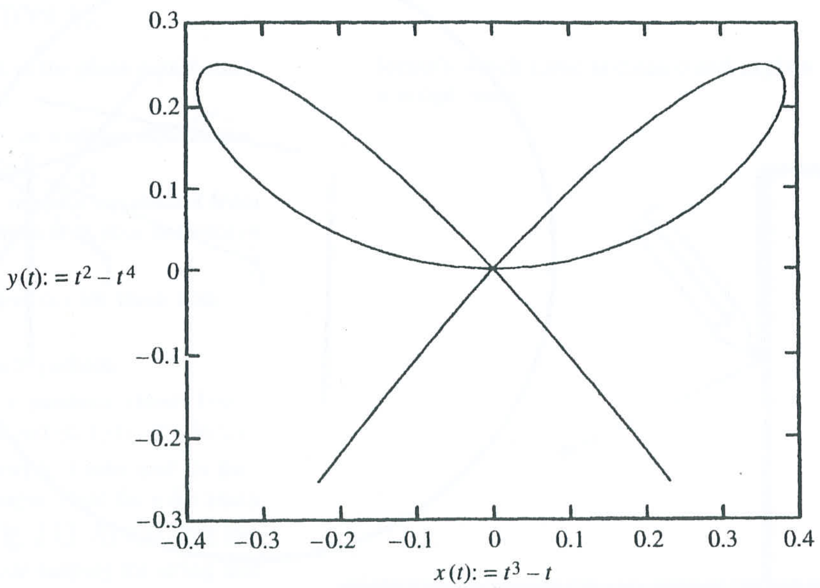


Figure 2.15

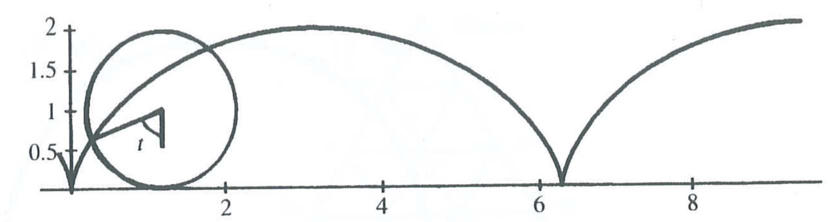


Figure 2.16 A cycloid.

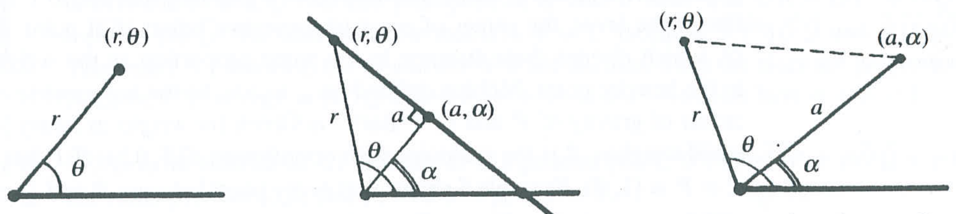
Exercise 1 (Calculus) For which values of t does the curve cross itself at $(0, 0)$? What happens to the curve when $x'(t) = 0$?

Example 2 A *cycloid* is the curve traced by a point on the circumference of a circle as the circle rolls along a line. Find parametric equations for a cycloid for a circle of radius r.

Solution. Let the point start at $(0, 0)$ when $t = 0$ and let a circle of radius 1 roll along the x-axis. Figure 2.16 illustrates a general point on this cycloid. As the circle rolls, the center moves along the line $y = 1$. When the circle has turned an angle of t radians, the center is at $(t, 1)$. The point rotates around the center, so its coordinates will be $(t + f(t), 1 + g(t))$, for some functions f and g. From trigonometry, we get $(t - \sin t, 1 - \cos t)$. Note that $1 - \cos t = 0$ only when t is a multiple of 2π, or when the circle has rolled a whole number of turns. How is the equation altered if the radius is r? ●

2.3.2 Polar coordinates

Jakob Bernoulli (1645–1705) developed polar coordinates, an alternative analytic model of Euclidean geometry (Fig. 2.17a). The first coordinate of a point gives its distance from the origin (positive or negative), and its second coordinate gives the angle made with the x-axis. A *point* in polar coordinates is an ordered pair of real numbers (r, θ), but two ordered pairs (r, θ) and (r', θ') can represent the same point if (1) $r = r' = 0$, (2) $r = r'$ and θ and θ' differ by a multiple of 360°, or (3) $r = -r'$ and θ and θ' differ by 180° or some odd multiple of 180°. A *line* in polar coordinates is a set of one of two forms: Lines through the origin satisfy $\theta = \alpha$ for some constant α; or lines whose closest point to the origin is (a, α) satisfy $r = a\sec(\theta - \alpha)$ for constants $a \neq 0$ and α (Fig. 2.17b). The law of cosines is used to show the *distance* between two points (r, θ) and (a, α) is $\sqrt{r^2 + a^2 - 2ra\cos(\theta - \alpha)}$, provided r and a are both positive (Fig. 2.17c).



a) Polar coordinates. b) The line with polar equation $r = a\sec(\theta - \alpha)$. c) Distance in polar coordinates $\sqrt{r^2 + a^2 - 2ra\cos(\theta - \alpha)}$.

Figure 2.17

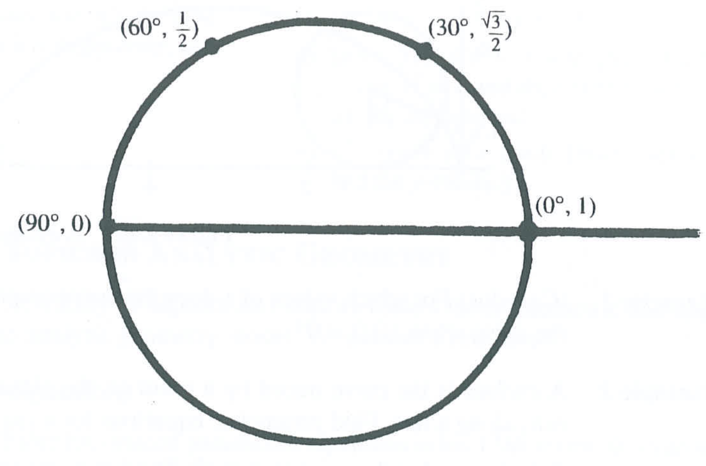


Figure 2.18 $r = \cos\theta$.

Example 3 The graph of $r = \cos\theta$ is the circle shown in Fig. 2.18. ●

2.3.3 Barycentric coordinates

Barycentric coordinates, devised by August Möbius in 1827, have a number of applications, including representing centers of gravity, which originally motivated Möbius. He later developed homogeneous coordinates in projective geometry from barycentric coordinates. (See Chapter 6.) Statisticians now use them in *trilinear plots*, as in Example 4.

Example 4 Figure 2.19 represents the election results of the 50 states and Washington, D.C. (the star), for the three-way presidential race in 1992. (We ignore other candidates for simplicity.) Three coordinates, the percentages, describe a state's outcome. The star for Washington, D.C., is at (86.3, 9.4, 4.3), the respective percentages for Clinton, Bush, and Perot. Two of the percentages determine the third, so the information is essentially two-dimensional. The dotted lines indicate where each candidate had a plurality. For example, the region for the winner, Clinton, is at the top with most of the dots. No dots are in Perot's region, indicating that he didn't win in any state. ●

Barycentric coordinates with two rather than three options are easier to use. Suppose that two points P and Q have weights w_p and w_q that add to 1. From Archimedes' law of the lever, the center of mass of these two points is at point R, between P and Q, which divides their distance in the same proportion as the weights with R closer to the heavier point. Möbius defined (w_p, w_q) to be the *barycentric coordinates* of the center of gravity of P and Q. ("Baro" is Greek for *weight* or *heavy*.) If P and Q have equal weights, R is the midpoint with coordinates (0.5, 0.5). If P has all of the weight, $R = P = (1, 0)$. Example 5 shows that every point between P and Q can be represented with barycentric coordinates. Example 6 justifies extending barycentric coordinates relative to three points. This same reasoning extends to any number of points in any number of dimensions. If the weights at the original points are $\alpha, \beta, \gamma, \ldots$ and add to 1, the center of gravity is $(\alpha, \beta, \gamma, \ldots)$. We can extend barycentric coordinates to points outside

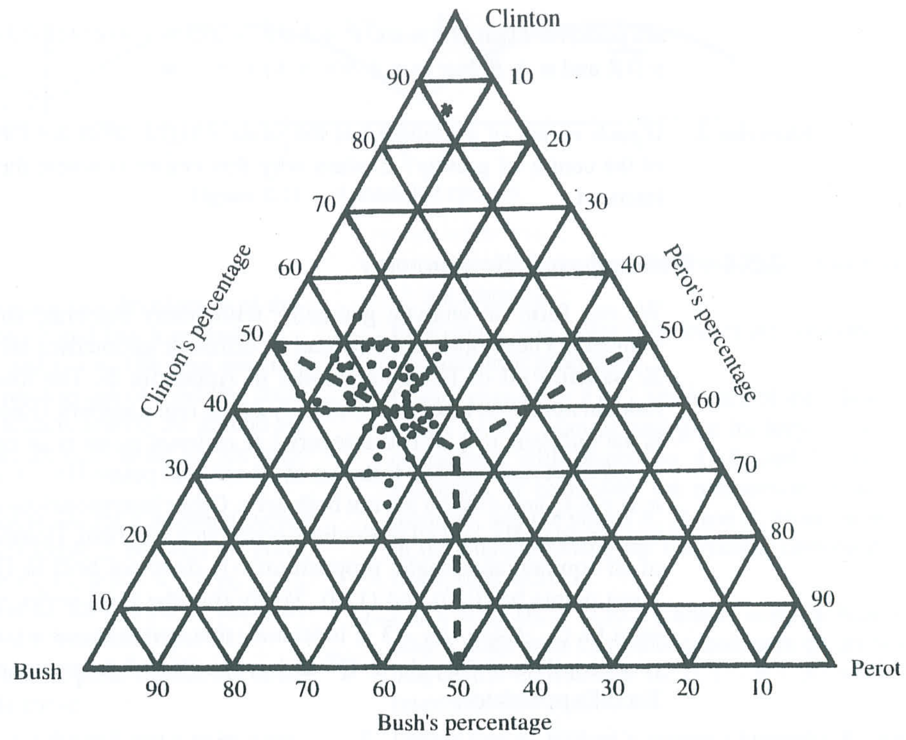


Figure 2.19 Trilinear plot of state and Washington, D.C., popular vote percentages in the 1992 presidential election.

the smallest convex set containing the original points, but some of the coordinates then will be negative.

Example 5 Let $\overrightarrow{AB}$ denote the vector from A to B and let O be any point not on $\overleftrightarrow{PQ}$. Then any point R between P and Q satisfies $\overrightarrow{OR} = \alpha\overrightarrow{OP} + \beta\overrightarrow{OQ}$, for some positive α and β with $\alpha + \beta = 1$. Then (α, β) are the barycentric coordinates of R with respect to P and Q.

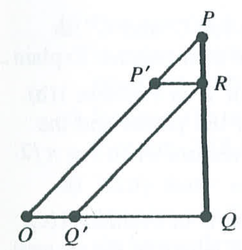


Figure 2.20

Solution. In Fig. 2.20 let $\overline{RP'} \| \overline{OQ}$ and $\overline{RQ'} \| \overline{OP}$. Then $\overrightarrow{OR} = \overrightarrow{OP'} + \overrightarrow{OQ'}$. As R is between P and Q, we can pick positive α and β such that $\overrightarrow{OR} = \overrightarrow{OP'} + \overrightarrow{OQ'} = \alpha\overrightarrow{OP} + \beta\overrightarrow{OQ}$. We need to show that $\alpha + \beta = 1$. Because $\overline{RP'} \| \overline{OQ}$ and $\overline{RQ'} \| \overline{OP}$, Theorem 1.5.1 shows $\triangle OPQ$ and $\triangle P'PR$ are similar. Thus the sides are proportional. Because $P'R = OQ' = \beta OQ$ and $OP' + P'P = OP$, we must have $\alpha + \beta = 1$. ●

Example 6 If S is in the interior of $\triangle PQR$ and O is a general point, $\overrightarrow{OS} = \alpha\overrightarrow{OP} + \beta\overrightarrow{OQ} + \gamma\overrightarrow{OR}$ for some positive α, β, and γ, with $\alpha + \beta + \gamma = 1$.

Solution. Draw a picture to illustrate this solution. Let T be the intersection of $\overleftrightarrow{QR}$ and $\overleftrightarrow{PS}$. By Example 5, $\overrightarrow{OS} = \alpha\overrightarrow{OP} + \delta\overrightarrow{OT}$, with $\alpha + \delta = 1$, and $\overrightarrow{OT} = \kappa\overrightarrow{OQ} + \lambda\overrightarrow{OR}$, with $\kappa + \lambda = 1$, and α, δ, κ, and λ are positive. Let $\beta = \delta \cdot \kappa$ and $\gamma = \delta \cdot \lambda$ so that both are positive. Then $\overrightarrow{OS} = \alpha\overrightarrow{OP} + \delta\overrightarrow{OT} = \alpha\overrightarrow{OP} + \delta(\kappa\overrightarrow{OQ} + \lambda\overrightarrow{OR}) = \alpha\overrightarrow{OP} + \beta\overrightarrow{OQ} + \gamma\overrightarrow{OR}$ and $\alpha + \beta + \gamma = \alpha + \delta(\kappa + \lambda) = 1$. ●

Exercise 2 If each vertex of a triangle has the same weight, what are the barycentric coordinates of the center of gravity? Explain why this center is where the medians of the triangle intersect.

2.3.4 Rational analytic geometry

We can form an analytic geometry with many algebraic structures besides the real numbers. These other models lead to different geometries because only the real plane $\mathbf{R}^2$ satisfies all of Hilbert's axioms in Appendix B. The model $\mathbf{Q}^2$ based on $\mathbf{Q}$, the rational numbers, loses the continuity of the real numbers. (Recall that a rational number is the quotient p/q of two integers.) A *rational point* is an ordered pair (x, y), where $x, y \in \mathbf{Q}$. A *rational line* is the set of all rational points $\{(x, y) : ax + by + c = 0\}$, where $a, b, c \in \mathbf{Q}$ and a and b are not both zero. Other interpretations remain the same as in the usual model, $\mathbf{R}^2$, and all of Euclid's postulates still hold. However, Euclid's construction of an equilateral triangle, proposition I-1, does not hold in $\mathbf{Q}^2$. Here, WLOG, let the given points be $(0, 0)$ and $(1, 0)$. Verify that the third vertex of the equilateral triangle must be $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. As $\sqrt{3}$ is irrational, this vertex is not a rational point. Hence there is no equilateral triangle in $\mathbf{Q}^2$ and proposition I-1, among others, is independent of Euclid's postulates.

Exercise 3 Explain why, if two lines with rational coefficients have a point of intersection, that intersection is a rational point in $\mathbf{Q}^2$.

(See a standard calculus text, such as Edwards and Penney [5, Chapters 10 and 12] for more on parametric equations and polar coordinates. Eves [6] has more varied information.)

PROBLEMS FOR SECTION 2.3

1. a) Explain why $C = \{(x, y) : x = \cos t, y = \sin t\}$ is a circle. Compare C and $C^* = \{(x, y) : x = \cos 2t, y = \sin 2t\}$.

b) Graph $B = \{(x, y) : x = \cos t, y = \sin 2t\}$, a figure 8.

c) Graph $S = \{(x, y) : x = t\cos t, y = t\sin t, t \geq 0\}$, a spiral.

d) Compare $D = \{(x, y) : x = f(t), y = g(t), p \leq t \leq q\}$, and $D^* = \{(x, y) : x = f(kt), y = g(kt), p/k \leq t \leq q/k\}$, where $k \neq 0$. [*Hint:* See part (a).]

2. (Calculus) The *tangent vector* to $(x(t), y(t))$ is $(x'(t), y'(t))$. The tangent vector's length $\|(x', y')\| = \sqrt{(x')^2 + (y')^2}$ is the speed of the point along the curve at time t. Assume that t is in radians.

a) Find the tangent vectors for C and C^* in Problem 1(a) and compare their lengths. Explain.

b) Find the tangent vector for B in Problem 1(b). Compare the positions of the points and the directions of the tangent vectors when $t = \pi/2$ and $t = 3\pi/2$.

c) Find the tangent vector for S in Problem 1(c). Compare the direction and length of the tangent vector when $t = 2\pi$ and when $t = 4\pi$. Describe the speed of the point along the spiral S as t increases.

d) Find the points and the tangent vectors for D and D^* in Problem 1(d) for $t = c$ and $t = c/k$, respectively. Compare these tangent vectors and their lengths.

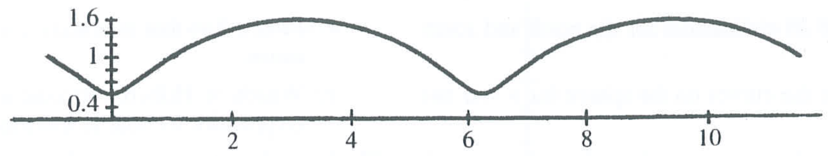


Figure 2.21 A modified cycloid.

3. (Calculus)

a) Find the tangent vector (see Problem 2) of the cycloid of Example 2 and use it to find the velocity at $t = 2\pi$ and $t = \pi$. Use your answer to explain why, in a photograph of a moving train, the spokes of the wheels nearest the ground are always in focus but the ones at the top are often blurred.

b) Find parametric equations for a modified cycloid similar to that in Example 2, but with the point inside the circle at $(0, k)$, $0 < k < 1$ (Fig. 2.21). Find the tangent vector for this curve. Compare the velocity at $t = 2\pi$ with that in part (a).

c) Repeat part (b) with the point outside the circle $(k < 0)$. Graph this curve.

4. Graph the curves having the following polar equations.

a) $r = k\theta$ (spiral).

b) $r^2 = \cos 2\theta$ (lemniscate of Bernoulli).

c) $r = 2\sin 3\theta$ (rose).

d) $r = 1 + \cos\theta$ (cartioid).

5. a) Devise formulas to convert Cartesian coordinates (x, y) to polar coordinates (r, θ) and vice versa.

b) Use part (a) to show how to write a function in polar coordinates $r = f(\theta)$ using parametric equations with $t = \theta$.

c) Use part (b) to verify that $r = \cos\theta$ is a circle. (See Fig. 2.18.)

d) Verify that $r = a\sec(\theta - \alpha)$, for $a \neq 0$, is the equation of a line. Find the Cartesian equation for the line with polar equation $r = \sqrt{2}\sec(\theta - 45°)$. Find the polar equation of the line having the Cartesian equation $\sqrt{3}x + y + 4 = 0$. [*Hint:* Use Fig. 2.17b), not part (a) of this problem.]

e) Use the law of cosines (Problem 3, Section 2.1) to justify the distance formula in polar coordinates. Explain how to modify that distance formula when one or both of the points' first coordinates are negative.

f) Explain why polar coordinates are a model of Euclidean geometry.

6. Draw a triangle ΔABC. Mark each of the following points on the triangle and give its barycentric coordinates with respect to A, B, and C: the midpoints of each side, the intersection of the medians, and the points A, B, and C. Mark the point in the triangle having the barycentric coordinates $(0.25, 0.25, 0.5)$.

7. Let $X = (1, 0)$, $Y = (0, 1)$, and $O = (0, 0)$. Explain why a point with Cartesian coordinates (α, β) has barycentric coordinates $(\alpha, \beta, 1 - \alpha - \beta)$ with respect to X, Y, and O.

8. Only a few of Hilbert's axioms (Appendix B) are false in the model $\mathbf{Q}^2$. Find them and explain why each fails in this model.

9. Consider the analytic geometry model $\mathbf{Z}^2$, where $\mathbf{Z}$ is the set of integers. Give suitable interpretations for Hilbert's undefined terms. Which of Hilbert's axioms (Appendix B) are true in the model $\mathbf{Z}^2$?

10. Assign coordinates to the points on a sphere, using their longitudes and latitudes (Fig. 2.22). Thus (x, y) represents a point, provided that $-90° \leq y \leq 90°$. Every point (x, y) has multiple representations, such as $(x + 360°, y)$.

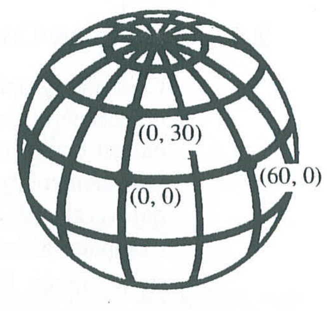


Figure 2.22 Coordinates on a sphere.

a) Describe all coordinates for the north and south poles.

b) Describe the curves on the sphere for $y = a$ and $x = b$.

c) Describe the curves on the sphere for $y = mx + b$.

d) If you interpret *point* as a point on the sphere and *line* as any of the curves of parts (b) and (c), which of Hilbert's axioms from group I (Appendix B) are true? Explain.

11. Assign coordinates to the points on the surface of a *torus* (a doughnut) using longitude and latitude measured in degrees (Fig. 2.23). Every point (x, y) has multiple representations, such as $(x + 360°, y + 720°)$. Interpret *point* as a point on the surface of a torus and *line* as the set of points satisfying an equation of the form $ax + by + c = 0$.

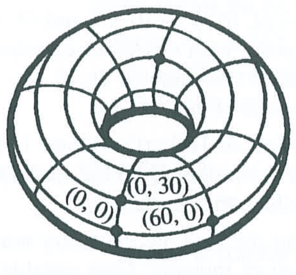


Figure 2.23 Coordinates on a torus.

a) Find values of a, b, and c so that the line $ax + by + c = 0$ is finitely long. Find other values so that that line is infinitely long.

b) Find a line that intersects n times with $y = 0$. Find a line that intersects $y = 0$ infinitely many times.

c) Which of Hilbert's axioms in group I and IV (Appendix B) hold in this model? Explain.

12. Taxicab geometry is an alternative analytic model with a distance function corresponding to the distances taxicabs travel on a rectangular grid of streets. The *taxicab distance* between $A = (a, b)$ and $S = (s, t)$ is $d_T(A, S) = |a - s| + |b - t|$. This geometry has quite different properties from Euclidean geometry. (See Section 1.4.)

a) How are two points that have the same Euclidean and taxicab distances related? If these distances are different, which is larger? If the smaller distance is 1, how large can the other distance be? Explain and illustrate.

b) A *taxicab circle* is the set of points at a fixed taxicab distance from a point. Describe taxicab circles. What are the possible types of intersection of two taxicab circles? Illustrate each.

c) A *taxicab midpoint* M of A and B satisfies $d_T(A, M) = d_T(M, B) = \frac{1}{2}d_T(A, B)$. Illustrate the different sets of midpoints two points can have.

d) The *taxicab perpendicular bisector* of two points comprises of all points equidistant from the two points. Describe and illustrate the different types of taxicab perpendicular bisectors.

e) In the plane find four points whose taxicab distances from each other are all the same.

13. Another distance formula on $\mathbf{R}^2$ uses the maximum of the x and y distances: For $A = (a, b)$ and $S = (s, t)$, $d_M(A, S) = Max\{|a - s|, |b - t|\}$. Redo Problem 12, using d_M.

2.4 Curves in Computer-Aided Design

Computer graphics depend heavily on analytic geometry. The computer stores the coordinates of the various points and information about how they are connected. The easiest connections are line segments and arcs of circles, which are easily described with elementary analytic geometry. However, connecting a series of points smoothly requires calculus and more advanced techniques, which we discuss briefly in this section. Computer-aided design (CAD) uses polynomials—a flexible and easily computed family of curves. For a sequence of points, we need a curve or a sequence of curves that smoothly passes through the points in the specified order. A simplistic approach finds one polynomial through all the points, as in the first two examples.