f) Vary this second approach to find the tangent to $y = 4x - x^2$ parallel to $y = 6x$.

g) Discuss any logical and practical shortcomings of this method.

9. a) Graph the following functions and decide which enclose a convex region of the plane: $y = x^2$, $y = x - x^2$, $y = x^3$, $y = x^4$, $y = e^x$, $y = \sin x$, and $y = \ln x$.

b) Functions f that satisfy $f((a+b)/2) \leq (f(a) + f(b))/2$ for all a and b are called *convex functions*. Which of the functions in part (a) are convex functions? How do convex functions compare with functions that enclose a convex region of the plane? To explain the difference between these uses of *convex* for functions, define *concave* functions. Explain how concave functions relate to functions enclosing a convex region.

c) (Calculus) Find the second derivative of the functions in part (a). What is special about the second derivative of the convex functions? Use a graph to explain how the definition of a convex function fits with what you found out about the second derivatives of convex functions. What can you say about the second derivative of the concave functions you defined in part (c)?

10. The arithmetic of complex numbers (**C**) has a well-known geometric interpretation in $\mathbf{R}^2$. The complex number $a + bi$ can be represented as the point, or vector, (a, b) in the plane. Addition of complex numbers corresponds to vector addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$.

a) Explain and illustrate on Cartesian axes why this addition satisfies the parallelogram law.

b) The *complex conjugate* of $a + bi$ is the number $a - bi$. Illustrate on Cartesian axes how these numbers are related geometrically.

c) The *modulus* of $a + bi$ is the real number $\sqrt{a^2 + b^2}$. What does the modulus tell you geometrically?

11. The formula for complex multiplication, $(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i$, doesn't reveal the geometry.

a) How does the product of a complex number and its conjugate relate to the modulus? (See Problem 10.)

b) Illustrate with several examples on Cartesian axes the result of multiplying $a + bi$ by a real number $r + 0i$. What corresponds geometrically to multiplying by a real number?

c) Illustrate on Cartesian axes the result of multiplying $a + bi$ by i, a complex number on the unit circle. Also illustrate the result of multiplying $a + bi$ by $0.6 + 0.8i$ and by $-0.96 + 0.28i$, other points on the unit circle. What do you think multiplication by a point on the unit circle does geometrically to $a + bi$?

d) Explain why any complex number $c + di$ can be written as the product of its modulus with a complex number $x + yi$ on the unit circle (for which $x^2 + y^2 = 1$). Use parts (b) and (c) to describe what multiplication by a general complex number $c + di$ does geometrically to $a + bi$.

2.2 Conics and Locus Problems

The Greeks identified and studied the three types of conics: ellipses, parabolas, and hyperbolas. However, nearly two thousand years passed before the first of many applications of conics outside of mathematics appeared. We call these curves conics because they are the intersections of a (double-napped) cone with planes at various angles (Fig. 2.6). To find the familiar equations of these curves we use an easier characterization based on distance. The process of finding a set of points or its equation from a geometric characterization is called a *locus problem*. See Eves [6] for further information on these topics.

Example 1 Find the set (locus) of points P such that P is the center of a circle tangent to two given lines.

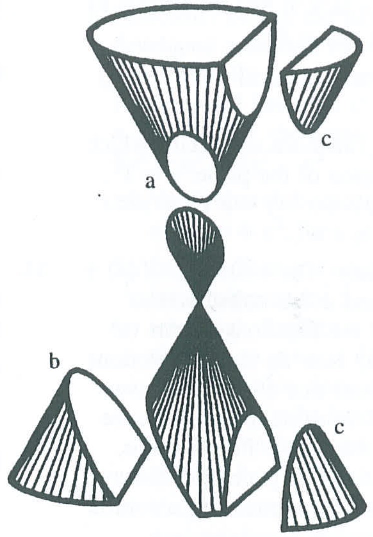

Figure 2.6 The intersection of a plane and a cone is (a) an ellipse, (b) a parabola, or (c) a hyperbola.

Solution. If the two lines are parallel, then a circle tangent to both must have its center on the line midway between these two lines. If the two lines intersect at a point Q, then the centers of circles tangent to both lines must be on one of the two angle bisectors of these two lines. Illustrate these two cases. ●

Definition 2.2.1 Given two points F and F', an *ellipse* is the set of points P in the plane such that the sum of the distances of P to F and F' is constant. Given two points F and F', a *hyperbola* is the set of points P in the plane such that the difference of the distances from P to F and F' is constant. Given a line m and a point F not on m, a *parabola* is the set of points P in the plane such that the distance of P from F equals the distance of P from m. The points F and F' are called *foci*. The line m is called the *directrix*. (Figs. 2.7, 2.8, and 2.9.)

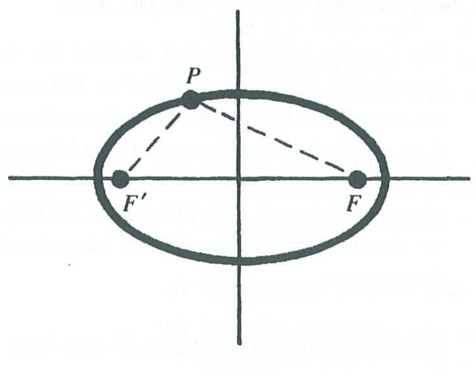

Figure 2.7 An ellipse.

PIERRE FERMAT

Although a lawyer and political councilor by profession, Pierre Fermat (1601?–1665) is best remembered for the mathematics he pursued in his spare time. He corresponded extensively with other mathematicians, rather than seeking publication. Indeed, his work in analytic geometry, even though done before Descartes, wasn't published until after Fermat died. He investigated the shapes of curves from their equations. This approach complemented Descartes's work, as Descartes had emphasized finding the equation of a curve defined by some geometric process. Fermat explicitly discussed first- and second-degree equations and explored many new curves with higher degree equations.

Fermat also contributed extensively and profoundly to number theory. He is best known for Fermat's last theorem: $a^n + b^n = c^n$ has no nontrivial integer solutions if $n > 2$. This statement holds the record for the greatest number of incorrect published "proofs," but Andrew Wiles finally proved it 1994. This simple-looking statement has led to extensive and profound investigations in number theory. Fermat's theorems in number theory concentrate on primes, divisibility, and powers. For example, Fermat's little theorem, an important tool in abstract algebra and coding theory, states that if a is not a multiple of a prime p, then p divides $a^{p-1} - 1$.

Other areas of mathematics also benefited from Fermat's creativity. The founding of probability as a mathematical subject grew out of letters between Fermat and Blaise Pascal. They corresponded at some length and eventually solved a problem on how to distribute fairly the wagers of an interrupted game of chance. Fermat was an important precursor of calculus with his method for finding maxima and minima. In addition, he understood the rules that we now describe as differentiating and integrating polynomials.

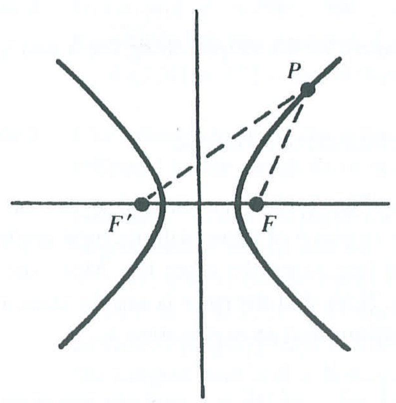

Figure 2.8 A hyperbola.

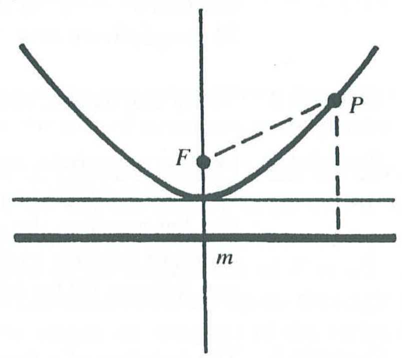

Figure 2.9 A parabola.

Exercise 1 Construct an ellipse as follows. Fix the two ends of a piece of string to different points on a piece of paper. Place a pencil on the paper so that it holds the string taut. Explain why the pencil sweeps out an ellipse as it moves.

Example 2 Find the equation of an ellipse.

Solution. WLOG, let the foci F and F' have coordinates $(f, 0)$ and $(-f, 0)$. Let $P = (x, y)$ be any point on the ellipse. The definition of an ellipse gives the following equation, which we adjust to a more familiar form.

$$\sqrt{(x-f)^2+y^2}+\sqrt{(x+f)^2+y^2}=k.$$

The sum of the distances is $k > 2f$.

$$\sqrt{(x-f)^2+y^2}=k-\sqrt{(x+f)^2+y^2}.$$

Square both sides and simplify.

$$x^2-2fx+f^2+y^2=k^2-2k\sqrt{(x+f)^2+y^2}+x^2+2fx+f^2+y^2.$$

$$-4fx-k^2=-2k\sqrt{(x+f)^2+y^2}.$$

Square again and simplify.

$$16f^2x^2+8fxk^2+k^4=4k^2(x^2+2fx+f^2+y^2).$$

$$x^2(16f^2-4k^2)-4k^2y^2=k^2(4f^2-k^2).$$

Divide by the right side.

$$\frac{4x^2}{k^2}+\frac{4y^2}{(k^2-4f^2)}=1.$$

We can rewrite this equation as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

because $k^2 - 4f^2 > 0$. The *diameters* of the ellipse along the x and y-axes are $2a$ and $2b$, respectively. ●

Exercise 2 Verify that a circle satisfies the definition of an ellipse.

Exercise 3 Outline a parabola, as indicated in Fig. 2.10. Fix a pin and a ruler on a piece of paper. Then place a right angled triangle or piece of paper with its right angle on the ruler and one leg touching the pin. Draw a line using the other leg. Move the triangle to draw different tangents to the parabola. Note that the ruler is not the directrix. (See Broman and Broman [2] for other constructions and an explanation.)

Example 3 The equation of a hyperbola is $x^2/a^2 - y^2/b^2 = 1$, and the equation of a parabola is $y = ax^2$.

Solution. See Problem 2. ●

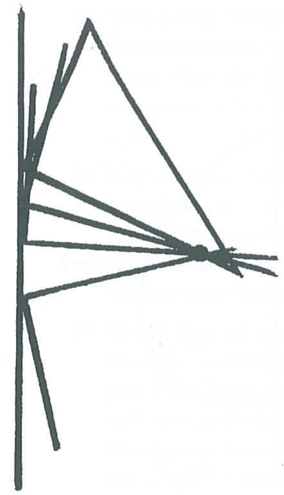

Figure 2.10

Examples 2 and 3 give the equations of the conics when we pick the easiest starting conditions. Less convenient conditions make the algebra more difficult and the final equation more complicated. The general equation of a conic is $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$. The conic is an ellipse, parabola, or hyperbola, depending on whether $ac - b^2$ is positive, zero, or negative, respectively. (Circles are special ellipses with $a = c$ and $b = 0$.) However, some of these general second-degree equations are *degenerate*: that is, they are equations for one or two lines or a point or even the empty set. The locus is a conic provided that the determinant of

$$\begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}$$

is nonzero.

Example 4 If we pick $F = (\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})$, $F' = (\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $k = 2\sqrt{2}$, as shown in Fig. 2.11, the ellipse has the equation $\frac{3}{4}x^2 + \frac{1}{2}xy + \frac{3}{4}y^2 = 1$. Note that $ac - b^2 = (3/4)(3/4) - (1/4)(1/4) = 1/2 > 0$ and that the determinant is $-1/2$. ●

Example 5 Use calculus to show the reflection property of the parabola: All rays from the focus are reflected by the parabola in rays parallel to the axis of symmetry of the parabola.

Solution. Turn the parabola horizontally, as shown in Fig. 2.12, and use calculus and trigonometry. Verify that the parabola $y = \pm\sqrt{4kx}$ has focus $F = (k, 0)$. For any point $P = (x, \sqrt{4kx})$ on the upper half of the parabola, show that the slope of the line from F to P is $\sqrt{4kx}/(x - k)$ and that the slope of the tangent line at P is $\sqrt{k}/\sqrt{x}$. The reflection property says that the angle between these two lines equals the angle between the tangent line and a horizontal line. The slopes are tangents of the various angles. Recall that

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+(\tan\alpha\tan\beta)}.$$

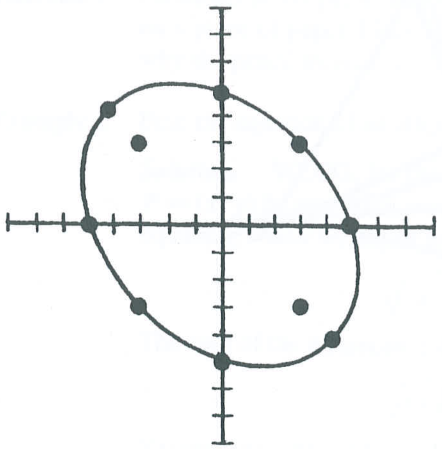

Figure 2.11 The ellipse $\frac{3}{4}x^2 + \frac{1}{2}xy + \frac{3}{4}y^2 = 1$.

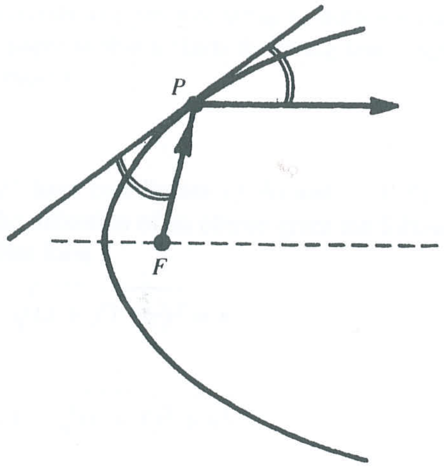

Figure 2.12 The reflection property of a parabola.

If $\tan\alpha = \sqrt{4kx}/(x-k)$ and $\tan(\beta) = \sqrt{k}/\sqrt{x}$, the right side of the formula for $\tan(\alpha - \beta)$ simplifies to $\sqrt{k}/\sqrt{x}$. It is the tangent for the angle between the tangent line and the horizontal line. Why is the lower half similar? The design of satellite disks makes use of this process in reverse: The disks take incoming parallel rays and focus them on one point. ●

PROBLEMS FOR SECTION 2.2

1. a) Find the locus of points in the plane equidistant from two points.

b) Find the locus of points on a sphere equidistant from two points.

c) Find the locus of points in space equidistant from two points. Relate this answer to your answers in parts (a) and (b).

d) Repeat parts (a), (b), and (c) for three non-collinear points.

2. a) Derive the equation of a hyperbola.

b) Derive the equation of a parabola. [*Hint:* Use $y = -k$ for the directrix and $(0, k)$ for the focus.]

3. a) Attach one end of a string to a ruler and fix the other end to a sheet of paper. Hold the ruler and a pencil as indicated in Fig. 2.13. As you slide the ruler back and forth, while keeping the string taut with the pencil, the pencil will trace out a curve. Identify which conic is created and explain why it is that conic.

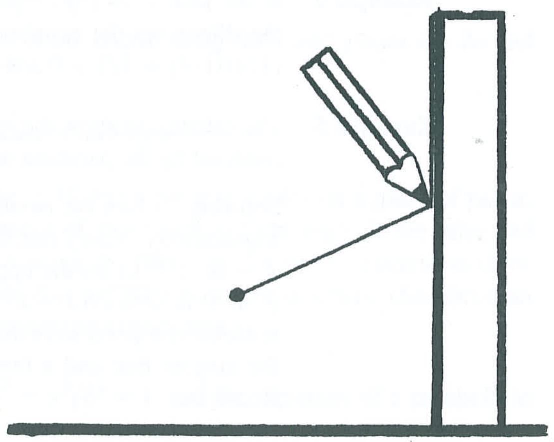

Figure 2.13

b) On wax paper or tracing paper draw a large circle and mark its center C and any other point A inside the circle. For various points P on the circle, fold and crease the paper so that points A and P coincide. The set of creases will outline an ellipse. Use the definition of an ellipse together with Fig. 2.14 to explain this result. Note that Q is a point on the ellipse.

c) Repeat part (b) with point A outside the circle. Modify Fig. 2.14 and explain this result.

4. a) Graph $y = \frac{1}{x}$, the most familiar equation for a hyperbola.

b) Rewrite this equation in the general form of a conic, verify that $ac - b^2 < 0$ and that the determinant is nonzero.

c) Verify that the foci of this hyperbola are at $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.

5. (Calculus) Every hyperbola has two *asymptotes*, or lines that the curve approaches but never intersects. A line $y = mx + b$ is an *oblique asymptote* of a function $y = f(x)$ provided that $\lim_{x\to\pm\infty} f(x) - (mx + b) = 0$.

a) Verify that the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ has asymptotes $y = (b/a)x$ and $y = -(b/a)x$. Show that the product of these two linear equations when rearranged gives a degenerate conic.

b) Use calculus to find the asymptotes of the hyperbola $y = x + (1/x)$. Verify that when you write this hyperbola's equation in the general form of a conic, $ac - b^2 < 0$ and that the determinant is nonzero.

c) Verify that all hyperbolas $y = x + (k/x)$, $k \neq 0$, have the same asymptotes as the hyperbola in part (b). Graph two hyperbolas, one with $k > 0$ and one with $k < 0$, on the same axes with the asymptotes. How does changing k change the graph? Show that every point not on the asymptotes is on just one of these hyperbolas.

6. Find and describe the locus of points P whose distance from a fixed point F is k times their distance from another fixed point F'. [*Hint:* Let $F = (k, 0)$ and $F' = (-1, 0)$. Consider the case $k = 1$ separately from other $k > 0$.]

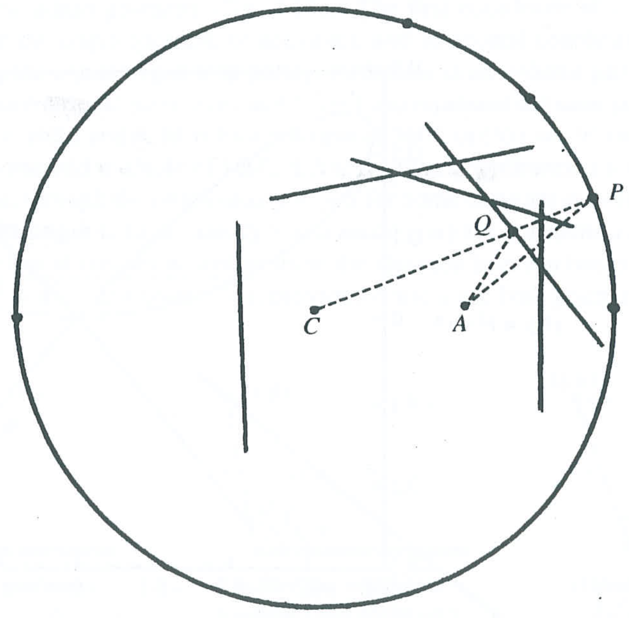

Figure 2.14

7. For each of the following equations, identify the type of conic it is, determine whether it is degenerate, and sketch its graph.
   a) $\frac{1}{2}y^2 + x - 2y = 0$.
   b) $\frac{1}{4}x^2 + y^2 - x - 6y + 9 = 0$.
   c) $\frac{1}{4}x^2 + y^2 - x - 6y + 10 = 0$.
   d) $2x^2 - 3xy + y^2 - 4 = 0$. [*Hint:* Factor $2x^2 - 3xy + y^2 = 0$ and then explain why the factors are the asymptotes.]
   e) $x^2 - xy + y^2 - 1 = 0$. [*Hint:* Pick x-values and find the y-values.]