2

Analytic Geometry

Computer graphics have greatly changed the designing of products. Eye-catching pictures and computing prowess easily attract attention, but the mathematics behind the pictures deserves fuller understanding. Analytic geometry helps represent the curves needed for a lawn mower part, an airplane, and other shapes. Transformational and projective geometries, the topics of Chapters 4 and 6, are the keys to enabling the computer show different views of a shape, including perspective.

> Though the idea behind it all is childishly simple, yet the method of analytic geometry is so powerful that very ordinary [youth] can use it to prove results which would have baffled the greatest of the Greek geometers—Euclid, Archimedes and Apollonius.
> —*E. T. Bell*

> How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality? —*Albert Einstein*

2.1 Overview and History

The fruitful union of algebra and geometry called analytic geometry has become an indispensable tool for mathematicians, scientists, and those in many other fields. Although René Descartes and Pierre Fermat deserve credit for creating analytic geometry, many others before and after shared in its development. (See Boyer [1].) After 150 B.C., Greek astronomers used coordinates to describe the positions of stars, and Greek and Roman geographers used coordinates to describe places on earth. In the fourteenth century Nicole Oresme used some examples of graphic representation. Arab and Renaissance mathematicians developed algebra into a powerful language. François Viète (1540–1603) used letters to represent unknown values and general situations. Viète called his approach *analysis*, following the ancient Greeks' meaning of the word. He started by assuming that the given problem had been solved and used a letter to represent this answer, which he then found algebraically in the modern sense. However, Viète retained the Greek limitation of adding only like quantities. Thus, in modern notation, Viète would be willing to add $x^3 + a^2x$, but not $x^3 + ax$, because the latter expression would represent a volume added to an area.

Pierre de Fermat (1601–1665) united the notational advances of Viète's algebra with traditional geometry. He realized that first- and second-degree equations correspond to lines and conics and investigated curves defined by higher degree equations. He solved some questions now considered part of calculus, such as finding maxima and minima. Although Fermat developed analytic geometry first, René Descartes (1596–1650) published sooner and was more influential. Descartes freed algebraic notation from Viète's restrictions of homogeneous dimensions. His famous book *Geometry*, published in 1637, showed the power of this new field, solving problems the Greeks had been unable to answer. Mathematicians began investigating the tremendous variety of curves suddenly described by algebraic equations. However, Descartes's book doesn't look like analytic geometry to us, for he didn't use coordinates and axes. Rather, he described the length of a line segment in terms of relationships of the lengths of various other line segments and translated these relationships into an algebraic equation. Nevertheless, we often call these coordinates Cartesian in honor of Descartes (and to distinguish them from other coordinate systems, such as polar coordinates).

During the first century of analytic geometry—and the early years of calculus—mathematicians didn't realize the power and simplicity of functions. Curves such as the folium of Descartes, shown in Fig. 2.1, were tackled as were more familiar curves such as the one shown in Fig. 2.2. Leonhard Euler (1707–1783), the most prolific mathematician of all time, emphasized functions and recast analytic geometry and calculus in nearly modern form in his influential textbooks.

Mathematicians have continued to develop analytic geometry and extend it into new branches of mathematics. In the nineteenth century mathematicians used analytic geometry to overcome visual limitations and investigate four and more dimensions. Transformational geometry and differential geometry grew out of analytic geometry, as well as areas no longer thought of as geometry, such as linear algebra and calculus. The advent of computer graphics has renewed the interest in analytic geometry.

RENÉ DESCARTES

René Descartes (1596–1650) won acclaim in philosophy, as well as in mathematics. In his most famous work, the *Discourse on the Method*, he tried to show how his rational approach could lead to new knowledge in any domain. He started by doubting everything that he did not know evidently to be true. His famous statement, "I think, therefore I am," was his first principle of philosophy, the idea that withstood all of his doubting. Next, he sought to reduce each difficulty he studied into many simpler parts and approach these parts systematically and exhaustively. This method may not seem revolutionary now, but Descartes was challenging traditional learning handed down for centuries. Descartes illustrated his method in three essays— *Optics, Geometry,* and *Meteorology*—that followed the *Method*. Of these, *Geometry* was by far the most influential.

In the first part of *Geometry*, Descartes showed how to solve a number of problems, including one that the Greeks had been unable to solve. He translated the geometric descriptions of curves into algebraic equations that he could then solve. In the second part, he provided a method of finding tangents to curves and attempted to classify curves. In the third part, he investigated the theory of equations, including his own rule of signs, giving a bound on the number of positive roots and what we call negative roots of a polynomial equation. (Descartes would have said that the equation $x + 5 = 0$ has 5 as a false root, rather than a root of -5.) (See Boyer [1] and Grabiner [7] for more on *Geometry*.)

Descartes intentionally wrote obscurely, making it difficult for others to extend his work. However, later editions of *Geometry* contained commentaries by others that explained his work, revealing the power of his method. The success of analytic geometry in posing and solving important problems has made it indispensable in mathematics. Of course, calculus depends crucially on analytic geometry, and it is no accident that so little time elapsed between Descartes's pioneering work in 1637 and the advent of calculus in the 1660s and 1670s.

2.1.1 The analytic model

We make the connection between geometric concepts and their algebraic counterparts explicit by building a model of geometry in algebra. The familiar graphs of analytic geometry are not actually part of the model, but they make the model and its many applications understandable. Geometric axioms and theorems become algebraic facts to be verified. In turn, algebraic equations and relations can be visualized.

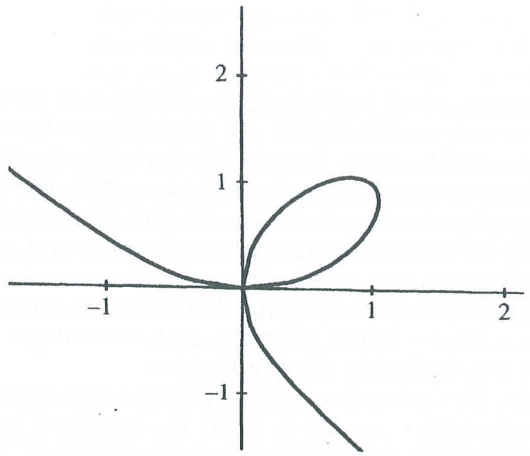

Figure 2.1 Folium of Descartes, $x^3 - 2xy + y^3 = 0$.

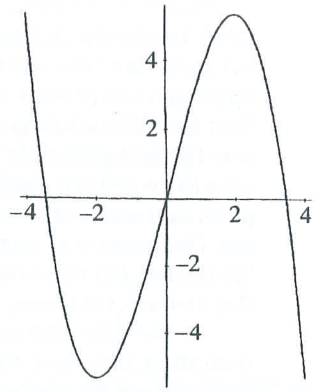

Figure 2.2 $y = \frac{1}{3}x^3 + 4x$.

Interpretation In the plane $\mathbf{R}^2$, by *point* we mean an ordered pair of real numbers (x, y). By *line* we mean a set of points of the form $\{(x, y) : ax + by + c = 0\}$, for $a, b, c \in \mathbf{R}$ with a and b not both 0. A point (u, v) is *on* the line $\{(x, y) : ax + by + c = 0\}$ iff $au + bv + c = 0$. The *distance* between two points $P = (x, y)$ and $Q = (u, v)$ is $d(P, Q) = \sqrt{(x-u)^2 + (y-v)^2}$.

Remarks As usual, we identify a line by its equation. Two lines $ax + by + c = 0$ and $mx + ny + p = 0$ are the same provided that there is a nonzero real number k such that $ak = m$, $bk = n$, and $ck = p$. The equation for the distance between two points given in the preceding interpretation is in essence the Pythagorean theorem—now no longer a theorem, but a definition.

Exercise 1 Find the slope and y-intercept of the line $ax + by + c = 0$, if $b \neq 0$. (When $b = 0$, the line is vertical and has no slope. If a and b are both 0, either no points or all points satisfy the equation $ax + by + c = 0$.)

Example 1 Verify that, for any two distinct points, there is only one line on both points. (This result shows that Hilbert's axioms I-1 and I-2 hold in this model.)

Verification. Let (x_1, y_1) and (x_2, y_2) be any two distinct points.

Case 1 $x_1 = x_2$. The line $x - x_1 = 0$ is on both points. Let $ax + by + c = 0$ be any line through these two points. Then $ax_1 + by_1 + c = 0$, and $ax_1 + by_2 + c = 0$. These equations reduce to $b(y_1 - y_2) = 0$. For the points to be distinct, $y_1 - y_2 \neq 0$. So $b = 0$, which forces $c = -ax_1$. Thus $ax + by + c = 0$ is a multiple of $x - x_1 = 0$, showing that only one line passes through these two points.

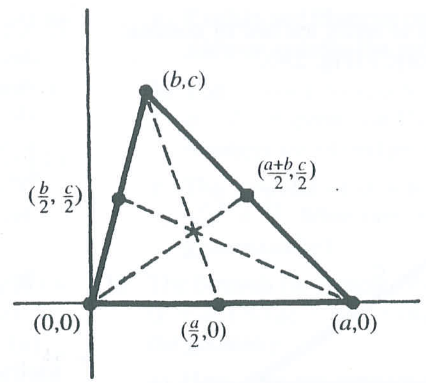

Figure 2.3

Case 2 $x_1 \neq x_2$. Verify $(y_2 - y_1/x_2 - x_1)x - y + y_1 - (y_2 - y_1/x_2 - x_1)x_1 = 0$ is a line on both points. As in case 1, verify that a line $ax + by + c = 0$ on both points is a multiple of the line given in this case. ●

Part of analytic geometry's power comes from our ability to solve geometric problems with algebra and to understand abstract algebraic expressions geometrically.

Example 2 Show that the medians of a triangle intersect in the point two-thirds the way from a vertex to the opposite midpoint.

Solution. WLOG, pick the axes so that the vertices of the triangle shown in Fig. 2.3 are $(0, 0)$, $(a, 0)$, and (b, c). Verify that the midpoints of the three sides are $(\frac{a}{2}, 0)$, $(\frac{b}{2}, \frac{c}{2})$, and $(\frac{a+b}{2}, \frac{c}{2})$. The *medians* connect midpoints to opposite vertices. Verify that the medians are $y = (c/(a+b))x$, $y = (c/(b-2a))x - ac/(b-2a)$ and $y = (2c/(2b-a))x - (ac/(2b-a))$. Verify that $(\frac{a+b}{3}, \frac{c}{3})$ is on all these lines and two-thirds the distance from each vertex to the opposite side. ●

(See Boyer [1] for more on the history of analytic geometry and Eves [6] for more on the model.)

PROBLEMS FOR SECTION 2.1

In these problems you may use familiar properties of geometry and analytic geometry such as: Nonvertical parallel lines have the same slope.

1. a) Guess what curve the midpoint of a ladder makes as the top of the ladder slips down a wall and the bottom of the ladder moves away from the wall. Draw a diagram.

b) Model the situation in part (a) with a ruler and a corner of a sheet of paper, marking the various midpoints of the ruler on the paper.

c) Use analytic geometry to find the set of all midpoints of segments of length 1 whose endpoints are on the x- and y-axes.

d) Explore what happens in part (b) if the corner of the paper doesn't form a right angle or if you pick a point on the ruler other than the midpoint.

2. Use analytic geometry to show that the four midpoints of any quadrilateral always form a parallelogram.

3. Use analytic geometry to verify the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos(C)$ (Fig. 2.4).

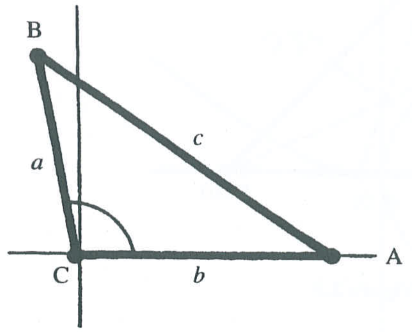

Figure 2.4 Law of cosines, $c^2 = a^2 + b^2 - 2ab\cos C$.

4. Verify Hilbert's axiom IV-1 (Appendix B): Through a given point P not on a given line k there passes at most one line that does not intersect k.

5. Define a circle in analytic geometry and verify Euclid's postulate 3 (Appendix A): To describe a circle with any center and radius.

6. For many applications, the use of different scales on the x- and y-axes is convenient (Fig. 2.5). Suppose that on the x-axis a unit represents a distance of k and on the y-axis a unit represents a distance of j. Explain why the equations of lines in this model have the same form as in the usual model. Develop a formula in this model for the distance between two points, $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. Give the equation of a circle of radius r and center (a, b).

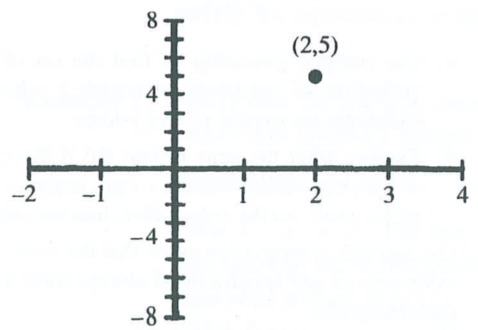

Figure 2.5

7. (Calculus) Before the advent of calculus, Fermat developed a method for finding the maximum or minimum of certain formulas, such as $bx^2 - x^3$. He reasoned first that if two values of x, say, u and v, give the same height, he would get $bu^2 - u^3 = bv^2 - v^3$.
   a) Verify that this equality reduces to $b(u + v) = u^2 + uv + v^2$, for $u \neq v$.
   b) Explain Fermat's reasoning that at a maximum (or minimum) for the formula the two x-values are equal. Replace v in part (a) with u and simplify to get $u = \frac{2}{3}b$. Verify, using calculus, that this value does indeed give a (relative) maximum for the function $y = bx^2 - x^3$. (Fermat didn't consider negative numbers or zero as answers.)
   c) Use Fermat's algebraic approach to find the (relative) maximum and minima for $x^4 - 2b^2x^2 + b^4$. Use calculus to verify that these values are correct.
   d) Explain any logical shortcomings of Fermat's approach.
   e) Explain the practical shortcomings of Fermat's method, by using $\cos(x) - \frac{1}{2}x$.

8. (Calculus) Most lines that intersect a curve do so in two (or more) places, say, a and b. When you solve the system of equations for the line and the curve, they reduce to $(x - a)(x - b)k = 0$, where k represents any other factors. However, tangents have a "double root" at the point of tangency: The system of a curve and its tangent line at a reduces to $(x - a)(x - a)k = 0$. You can use this idea to find tangents without calculus. First, find the tangent to $y = x^2 - x$ at $(2, 2)$.
   a) Verify that the equation of a general line through $(2, 2)$ is $y = mx - 2m + 2$.
   b) Substitute $mx - 2m + 2$ for y in $y = x^2 - x$ and find x in terms of m with the aid of the quadratic formula.
   c) Which value of m in part (b) gives one value of x (a double root)? [*Hint:* Consider what is under the $\sqrt{\ }$.) Use this m to find the tangent's equation.]
   d) Use calculus to verify your answer in part (c).
   e) Vary the preceding approach to find the two tangents to $y = x^2 - 2x + 4$ through the point $(0, 0)$. Graph this parabola and these tangents.
   f) Vary this second approach to find the tangent to $y = 4x - x^2$ parallel to $y = 6x$.
   g) Discuss any logical and practical shortcomings of this method.

9. a) Graph the following functions and decide which enclose a convex region of the plane: $y = x^2$, $y = x - x^2$, $y = x^3$, $y = x^4$, $y = e^x$, $y = \sin x$, and $y = \ln x$.
   b) Functions f that satisfy $f((a + b)/2) \leq (f(a) + f(b))/2$ for all a and b are called *convex functions*. Which of the functions in part (a) are convex functions? How do convex functions compare with functions that enclose a convex region of the plane? To explain the difference between these uses of *convex* for functions, define *concave* functions. Explain how concave functions relate to functions enclosing a convex region.
   c) (Calculus) Find the second derivative of the functions in part (a). What is special about the second derivative of the convex functions? Use a graph to explain how the definition of a convex function fits with what you found out about the second derivatives of convex functions. What can you say about the second derivative of the concave functions you defined in part (c)?

10. The arithmetic of complex numbers (**C**) has a well-known geometric interpretation in $\mathbf{R}^2$. The complex number $a + bi$ can be represented as the point, or vector, (a, b) in the plane. Addition of complex numbers corresponds to vector addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
   a) Explain and illustrate on Cartesian axes why this addition satisfies the parallelogram law.
   b) The *complex conjugate* of $a + bi$ is the number $a - bi$. Illustrate on Cartesian axes how these numbers are related geometrically.
   c) The *modulus* of $a + bi$ is the real number $\sqrt{a^2 + b^2}$. What does the modulus tell you geometrically?

11. The formula for complex multiplication, $(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i$, doesn't reveal the geometry.
   a) How does the product of a complex number and its conjugate relate to the modulus? (See Problem 10.)
   b) Illustrate with several examples on Cartesian axes the result of multiplying $a + bi$ by a real number $r + 0i$. What corresponds geometrically to multiplying by a real number?
   c) Illustrate on Cartesian axes the result of multiplying $a + bi$ by i, a complex number on the unit circle. Also illustrate the result of multiplying $a + bi$ by $0.6 + 0.8i$ and by $-0.96 + 0.28i$, other points on the unit circle. What do you think multiplication by a point on the unit circle does geometrically to $a + bi$?
   d) Explain why any complex number $c + di$ can be written as the product of its modulus with a complex number $x + yi$ on the unit circle (for which $x^2 + y^2 = 1$). Use parts (b) and (c) to describe what multiplication by a general complex number $c + di$ does geometrically to $a + bi$.

2.2 Conics and Locus Problems

The Greeks identified and studied the three types of conics: ellipses, parabolas, and hyperbolas. However, nearly two thousand years passed before the first of many applications of conics outside of mathematics appeared. We call these curves conics because they are the intersections of a (double-napped) cone with planes at various angles (Fig. 2.6). To find the familiar equations of these curves we use an easier characterization based on distance. The process of finding a set of points or its equation from a geometric characterization is called a *locus problem*. See Eves [6] for further information on these topics.

Example 1 Find the set (locus) of points P such that P is the center of a circle tangent to two given lines.