1.6 Three-Dimensional Geometry

Although we live in a three-dimensional world, visualizing three-dimensional geometric figures is harder than visualizing two-dimensional figures. We explore nonaxiomatically the geometry of polyhedra (the plural of polyhedron) and the sphere to help you deepen your visual intuition. Working with physical models aids this understanding beyond what any textbook figures can do.

1.6.1 Polyhedra

Polyhedra continue to fascinate people, much as they did the ancient Greeks. We assume an intuitive understanding of a polyhedron because an exact definition is more complicated than our treatment warrants. (See Lakatos [20].) In brief, a *polyhedron* is composed of *vertices*, *edges*, and *faces*. The faces are polygons. At each vertex (or corner) at least three faces meet, and at each edge (line segment) two faces meet.

Pyramids are easy to visualize, and all polyhedra can be dissected into pyramids, much as all polygons can be dissected into triangles. A *pyramid* has a polygon with

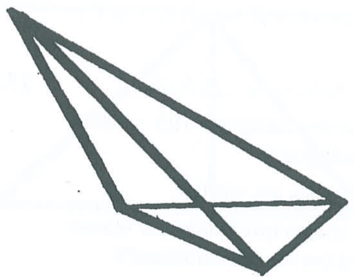

Figure 1.42

n edges for its *base*, one more vertex not in the plane of the base called the *apex*, and n triangular faces that are determined by the apex and the n edges of the base. The polyhedron with the fewest vertices, edges and faces is a triangular pyramid or *tetrahedron*, shown in Fig. 1.42.

Exercise 1 Verify that a pyramid with an n-gon for a base has $V = n + 1$ vertices, $E = 2n$ edges, and $F = n + 1$ faces.

Example 1 Use calculus to find a formula for the volume of a pyramid.

Solution. WLOG select a coordinate axis so that the origin is at the apex of the pyramid and the x-axis is perpendicular to the base (Fig. 1.43). Suppose that the area of the base is B and that the x-coordinate of points on the base is $h > 0$, the height of the pyramid. Any cross section of the pyramid in a plane parallel to the base is a polygon similar to the base. If a cross section's x-coordinate is x, then Problem 12 of Section 1.5 shows its area to be $(x/h)^2B$. Thus the volume of the pyramid is

$$\int_0^h \left(\frac{x}{h}\right)^2 B dx = \int_0^h \frac{x^2}{h^2} B dx = \left(\frac{x^3}{3h^2} B\right)_0^h = \frac{1}{3} h B.$$

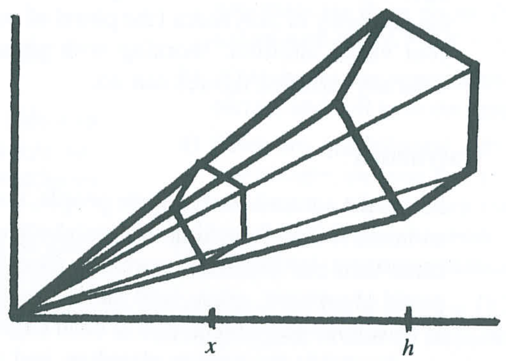

Figure 1.43

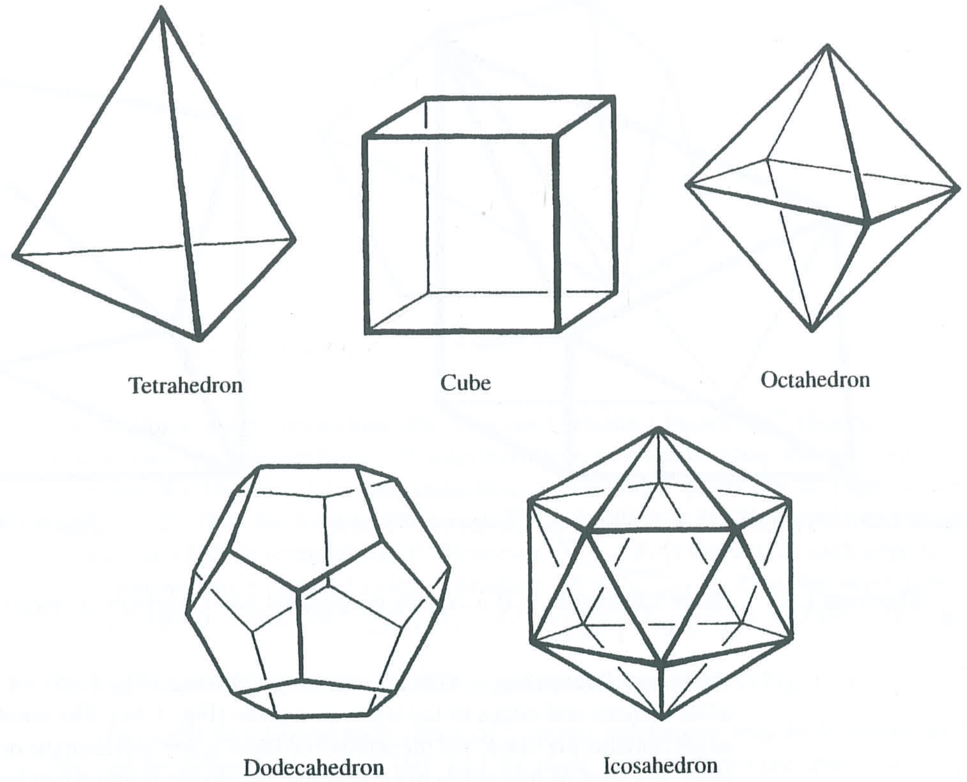

Figure 1.44 The regular polyhedra.

Exercise 2 Modify Example 1 to find the formula for the volume of a cone.

Exercise 3 Explain how, in principle, you can use pyramids to find the volume of any polyhedron.

The five polyhedra shown in Fig. 1.44 possess regular properties and a high degree of symmetry. These regular polyhedra are often called the Platonic solids because the Greek philosopher Plato was fascinated by them and his discussion of them is the oldest that survives. A convex polyhedron is *regular* provided that all its faces are the same regular polygon and the same number of polygons meet at each vertex. (See Coxeter [5, Chapter 10] for more information.)

The noted Swiss mathematician Leonhard Euler (1707–1783) developed a formula relating the number of vertices, edges, and faces for a large collection of polyhedra, including all convex ones. Because a careful proof, even for convex polyhedra, would require an overly long and technical development, we don't present one here. (See Beck et al [2] and Lakatos [20].) Euler's Formula has a wide variety of applications in geometry, graph theory, and topology.

Exercise 4 Find the number of vertices, V, edges, E, and faces, F, for the regular polyhedra in Fig. 1.44.

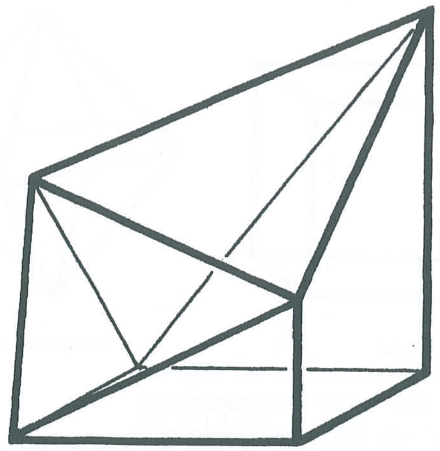

Figure 1.45 A polyhedron with 7 vertices, 13 edges, and 8 faces.

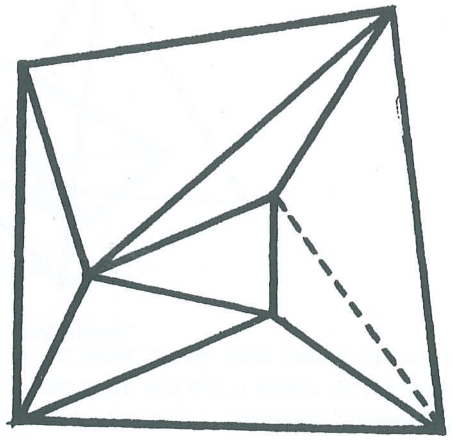

Figure 1.46

Theorem 1.6.1 **Euler's Formula** If a convex polyhedron has V vertices, E edges and F faces, then $V - E + F = 2$.

Outline of Reasoning. Given a convex polyhedron (Fig. 1.45) we can stretch the net of its vertices and edges to lay it out on a plane (Fig. 1.46). The number of vertices and edges remains the same, but the number of faces is one less than the original polyhedron. Next, if a face of this net is not a triangle, we divide it into triangles by adding edges. A technical argument shows that this division can always be made and that doing so increases the number of faces and edges the same amount. Another argument shows that we can carefully eliminate edges one at a time, each time eliminating either a face or a vertex. Thus we preserve the value $V - E + F$. (Try this process with Fig. 1.46.) In the end, we are left with a triangle, for which we have $V - E + F = 3 - 3 + 1 = 1$. Thus the original polyhedron must satisfy Euler's formula. ■

Example 2 The angle sum of all the angles of a tetrahedron is $4 \times 180° = 720°$ because a tetrahedron has four triangles for faces. Similarly the six square faces of a cube give an angle sum of $6 \times 360° = 2160°$. A simple formula for the angle sum of all convex polyhedron depends on shifting our focus from the faces to the vertices, as Descartes discovered. ●

Theorem 1.6.2 **Descartes's Formula** If a convex polyhedron has V vertices, then the angle sum of all of the angles around all the vertices is $360°(V - 2)$.

Proof. We derive this result from Euler's formula. We rewrite $V - E + F = 2$ as $V - 2 = E - F$. Note that the left side multiplied by 360 is Descartes's formula: $360°(V - 2) = 360°(E - F) = 180°(2E - 2F)$. Next we relate the right side to the angle sum.

Let F_i be the number of faces with i edges on them; for example, F_3 is the number of triangles. The number of faces is $F = F_3 + F_4 + F_5 + \cdots = \sum_{i=3}^{\infty} F_i$. (Of course, for a given polyhedron, the sum isn't infinite.) Figure 1.47 illustrates that a convex face with i edges has $i - 3$ diagonals from a vertex, dividing the face into $i - 2$ triangles. By Theorem 1.1.1 and induction, a face with i edges has an angle sum of $180°(i - 2)$. Thus the total angle sum is $\sum_{i=3}^{\infty} F_i 180°(i - 2)$. We can also count the number of edges in terms of the F_i. However, $\sum_{i=3}^{\infty} iF_i$ is too big, as each edge is on two faces and so is counted twice. Therefore $2E = \sum_{i=3}^{\infty} iF_i$. Thus the angle sum is $\sum_{i=3}^{\infty} F_i 180°(i - 2) = 180° \sum_{i=3}^{\infty} iF_i - 2F_i = 180°(2E - 2F) = 360°(V - 2)$. ■

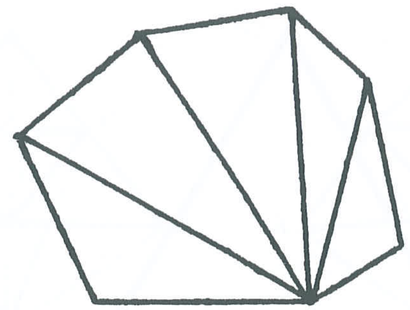

Figure 1.47

Exercise 5 Verify Descartes's formula for the five regular polyhedra shown in Fig. 1.44.

The angle sum around each vertex of a convex polyhedron must be less than 360°. We call the difference between 360° and the angle sum at a vertex the *spherical excess* of that vertex. Descartes's formula tells us that the total spherical excess of all vertices is 720°.

1.6.2 Geodesic domes

Buckminster Fuller (1895–1983) related the concept of spherical excess to the strength of geodesic domes, which he invented. He was a prolific inventor who devoted years to designing economical, efficient buildings. Traditional buildings, designed as modified rectangular boxes, require the use of a lot of material to enclose a given volume and support a roof. In contrast, a sphere, the shape that minimizes the surface area for a given volume, is expensive to build because of its curved surface. Fuller avoided these drawbacks of the box and the sphere by starting with an icosahedron. The domed effect of the icosahedron distributes the weight of the structure evenly, as spherical domes do, without needing a curved surface. Furthermore, the triangular faces of the icosahedron are structurally stronger than the rectangular faces of traditional buildings. Fuller was able to increase building size by dividing each of the icosahedron's 20 faces into smaller triangles. To maximize the strength of the building, he found that he needed to arrange these smaller triangles so that all the vertices were on the surface of a sphere. He coined the name *geodesic dome* for a convex polyhedron whose faces are all triangles.

We consider only domes based on an icosahedron. The *frequency* of a dome is n, where each of the original triangles is divided into n^2 smaller triangles (Fig. 1.48). To determine a geodesic dome we need to know the measures of all the edges and angles.

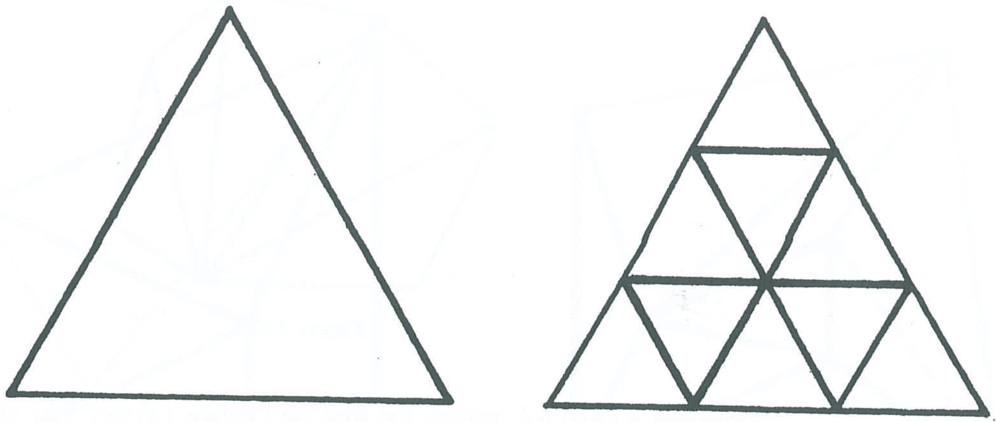

Figure 1.48 The division of one face of an icosahedron to make a three-frequency dome.

As Example 3 illustrates, such three-dimensional calculations rely extensively on two-dimensional geometry.

Example 3 Find the edge lengths and angle measures for the two-frequency dome shown in Fig. 1.49, if the radius OA of the sphere is 1.

Solution. From Problem 6 of this section we know that the length $AB \approx 1.05146$. Because D' and E' are the midpoints of $\overline{AB}$ and $\overline{AC}$, $AD' = D'E' \approx 0.52573$. By the Pythagorean theorem, $OD' \approx 0.85065$. The ray $\overrightarrow{OD'}$ intersects the sphere at D and, again by the Pythagorean theorem, $AD \approx 0.54653$. By Theorem 1.5.4 the triangles $\triangle OD'E'$ and $\triangle ODE$ are similar. Hence $DE \approx 0.61803$. Next we can use the law of cosines, $c^2 = a^2 + b^2 - 2ab\cos(C)$, to determine that $m\angle DAE \approx 68.86°$. Theorem 1.1.1 gives us 55.57° for $m\angle ADE$ and $m\angle AED$. The angles of the other corner triangles are the same, and the angles of the center triangle $\triangle DEF$ are all 60° because it is equilateral. ●

The strength of an icosahedron relies on two facts. First, triangular faces are structurally stable on their own, a consequence of the congruence theorem SSS. Second, these triangular faces distribute forces well because of the angles where they meet. One good measure of these angles is the spherical excess at each vertex. For an icosahedron, the spherical excess at each vertex is 60°. As the number of vertices increases with the frequency of the dome, Descartes's formula necessitates a decrease in the spherical excesses at the vertices. However, the proper design of the dome maximizes the smallest of these spherical excesses and so maximizes strength. Such a design will also make the triangles roughly equilateral, distributing the weight of the dome better and so strengthening the dome. High-precision technology and modern materials enable domes to be structurally stable with spherical excesses as small as $\frac{1}{2}^{\circ}$.

Exercise 6 For the two-frequency icosahedron of Example 3, show that the spherical excesses of the two kinds of vertices are 15.7° and 17.72°.

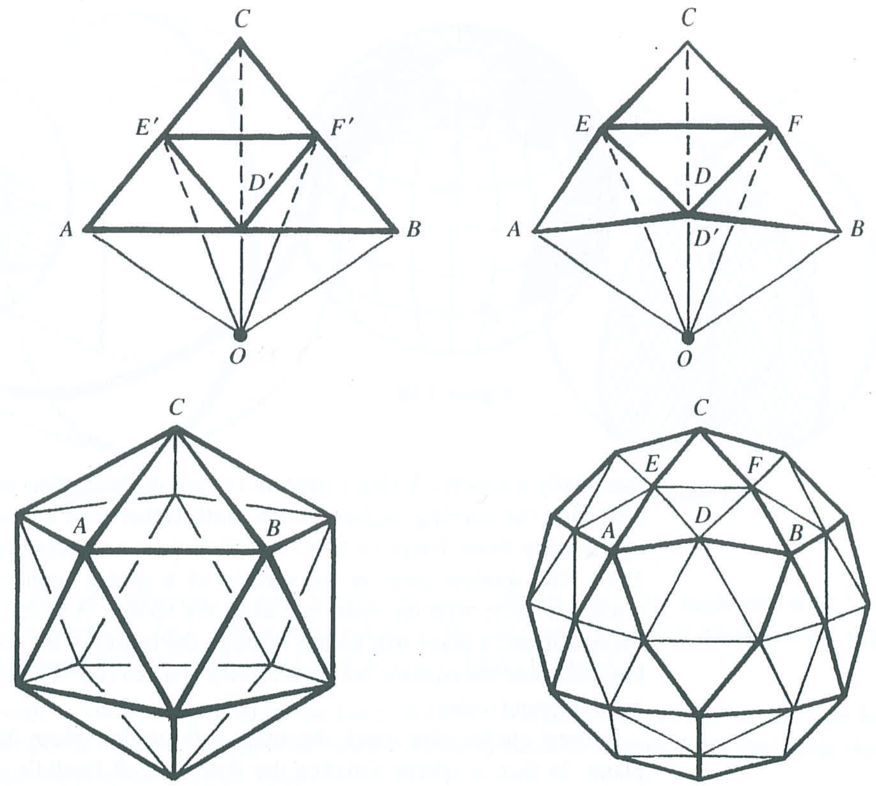

Figure 1.49 Each face ($\triangle ABC$) of the icosahedron becomes four faces on the two-frequency dome. Points D, E, and F are on OD', OE', and OF' and the same distance from O as A, B, and C are.

The strength of geodesic domes is extraordinary. The U.S. Air Force tested the strength of a 55-foot diameter fiber glass geodesic dome in 1955 before choosing domes to house the radar antenna of the Distant Early Warning system. They linked the vertices of the dome to a winch connected to a 17-ton concrete slab buried under the dome. They had intended to tighten the winch until the dome collapsed under the strain and then measure the breaking point. However, instead of collapsing, the dome withstood the stress and actually lifted the concrete slab. The great strength of geodesic domes enables them to enclose large spaces without interior supports. The largest geodesic dome, over 400 feet in diameter, far surpasses the largest space without interior support of any conventional building. (See Edmondson [8] and Kenner [17] for more on Fuller and geodesic domes.)

1.6.3 The geometry of the sphere

The needs of astronomy and global navigation have prompted the study of the geometry of the sphere for centuries. Astronomers use the inside surface of a sphere to describe the positions of the stars, planets, and other objects. For navigational purposes, the earth is

Figure 1.50

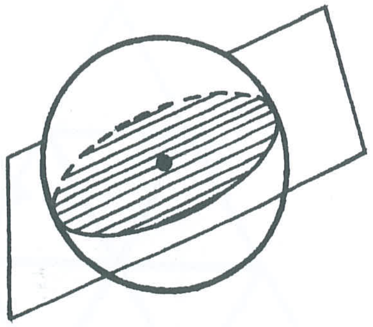

Figure 1.51

essentially a sphere. A ship's captain (or an airplane pilot) seeks the most direct route, following the curving surface of the earth rather than an Euclidean straight line. The shortest trip from Tokyo to San Francisco goes considerably north of either city (Fig. 1.50). The shortest path on the surface of a sphere connecting two points is a *great circle*—a circle with the same radius as the sphere. A great circle is the intersection of the sphere and a plane that passes through the center of the sphere (Fig. 1.51). Circles of longitude and the equator are great circles. The circles of latitude, except for the equator, are not great circles.

Great circles play much the same role on the sphere that straight lines do on the plane. In fact, a sphere satisfies the first four of Euclid's postulates in Appendix A. However, the geometric properties of lines and great circles differ in important ways. There are no parallel great circles because two distinct great circles always intersect in two diametrically opposed points. We use the same letter to denote them, placing a prime on one of them to distinguish one from the other. Two great circles divide the sphere into four regions called *lunes* (Fig. 1.52). The angle that these two great circles make is the angle at which the two planes meet (Fig. 1.53). Three great circles with no common point of intersection form eight *spherical triangles* (Fig. 1.54).

Exercise 7 Use Fig. 1.53 to explain why the area of a lune with an angle of $\alpha°$ is $(\alpha/90)\pi r^2$. Recall that the surface area of a sphere of radius r is $4\pi r^2$.

Exercise 8 Suppose that a spherical triangle $\triangle ABC$ has angles measuring 70°, 80°, and 90°. Verify that the angle sums for the seven related spherical triangles, such as $\triangle ABC'$, also are more than 180°.

In Theorem 1.6.3, we show not only that the angle sum of a spherical triangle is more than 180°, but also that this angle sum is related to the triangle's area. The spherical triangle $\triangle ABC$ shown in Fig. 1.54 determines three overlapping lunes $BACA'$, $ABCB'$, and $ACBC'$. The opposite spherical triangle $\triangle A'B'C'$ determines three other lunes that do not intersect these three lunes for $\triangle ABC$. Together the six lunes cover the entire sphere.

Figure 1.52

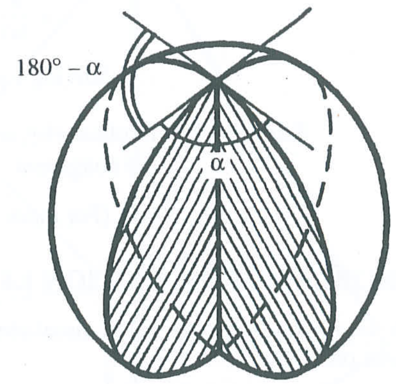

Figure 1.53

Theorem 1.6.3 The area of a spherical triangle is proportional to the excess of its angle sum over 180°. More precisely, on a sphere with radius r, the area of a spherical triangle with angle measures of $\alpha°$, $\beta°$, and $\gamma°$ is $[(\alpha+\beta+\gamma-180)/180]\pi r^2$.

Proof. The area covered by the lunes $BACA'$, $ABCB'$, and $ACBC'$ is $2\pi r^2$, or half the sphere because the opposite lunes cover a symmetric region. Note that these three lunes each cover $\triangle ABC$. Then

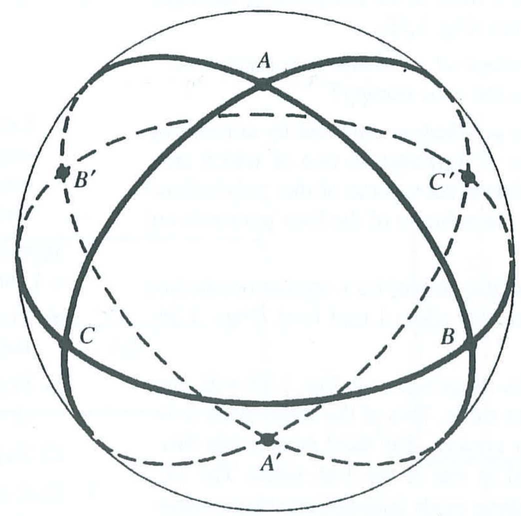

Figure 1.54

$$2\pi r^2 = \text{Area}(BACA') + \text{Area}(ABCB') + \text{Area}(ACBC') - 2 \cdot \text{Area}(\triangle ABC)$$

$$= \frac{\alpha}{90}\pi r^2 + \frac{\beta}{90}\pi r^2 + \frac{\gamma}{90}\pi r^2 - 2 \cdot \text{Area}(\triangle ABC).$$

Solving for Area($\triangle ABC$) gives $[(\alpha + \beta + \gamma - 180)/180]\pi r^2$. ■

Exercise 9 Explain why, in spherical geometry, if two triangles on the same sphere are similar, they are congruent.

(For more information on spherical geometry, see McCleary [21, 3ff].)

PROBLEMS FOR SECTION 1.6

Physical models greatly aid in visualizing and solving these problems.

1. a) Construct models of the five regular polyhedra. (See Fig. 1.44 or Wenninger [28].)

b) Prove that there are only five regular polyhedra, using spherical excess. [*Hint:* Why do regular polygons with more than five sides not need to be considered? Find the largest number of regular pentagons that can fit around a vertex. Repeat for squares and equilateral triangles.]

2. For a rectangular box with sides a, b, and c, explain why the diagonal d satisfies the "three-dimensional Pythagorean theorem" $a^2 + b^2 + c^2 = d^2$.

3. Suppose that the edge of a cube is 1 unit long.

a) Find the distances from the center of the cube to the center of a face, to the midpoint of an edge, and to a vertex (Fig. 1.55).

b) What percentage of a circumscribed sphere's volume does the cube occupy?

c) Describe the polyhedron obtained by connecting four vertices of the cube, no two of which are adjacent. What is the volume of this polyhedron? [*Hint:* Find the volumes of the four pyramids cut away.]

4. Repeat Problem 3(a) and (b) for a regular octahedron and tetrahedron with edges 1 unit long (Figs. 1.56 and 1.57).

5. a) Construct the shape shown in Fig. 1.58 with three 3×5-in. note cards. Two of the cards need 3-in. slits in their centers. The third card needs this slit extended to one of the 3-in. sides. The 12 corners of these cards approximate the vertices of a regular icosahedron.

b) Modify part (a) to find the exact coordinates of the vertices of a regular icosahedron as follows.

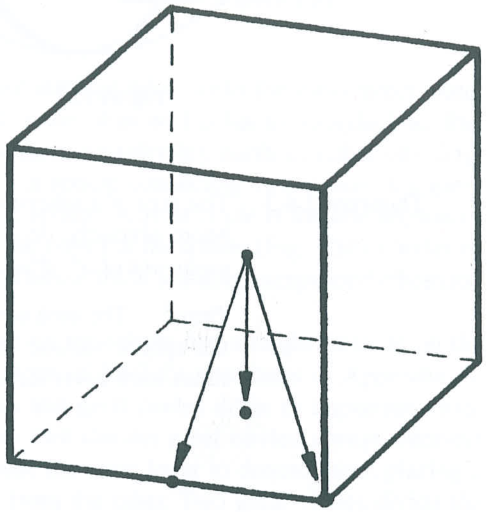

Figure 1.55

Let the slits be on the x-, y-, and z-axes and let the dimensions of the modified cards be $2 \times 2b$. Why are the 12 vertices at $(\pm 1, \pm b, 0)$, $(0, \pm 1, \pm b)$, and $(\pm b, 0, \pm 1)$? Find b.

6. Suppose that the edge $\overline{AB}$ of a regular icosahedron is 1 unit long (Fig. 1.59).

a) Find the lengths of the "short diagonal" $\overline{AC}$ and the "long diagonal" $\overline{BC}$. (See Problem 5.)

b) Find the length of the edge of a regular icosahedron inscribed in a sphere of radius 1.

c) Repeat Problem 3(a) and (b) for the icosahedron.

7. Find eight vertices of a regular dodecahedron that are the vertices of a cube. (See Fig. 1.44.)

8. The *excess* of a spherical triangle is the difference between its angle sum and 180°. Without using

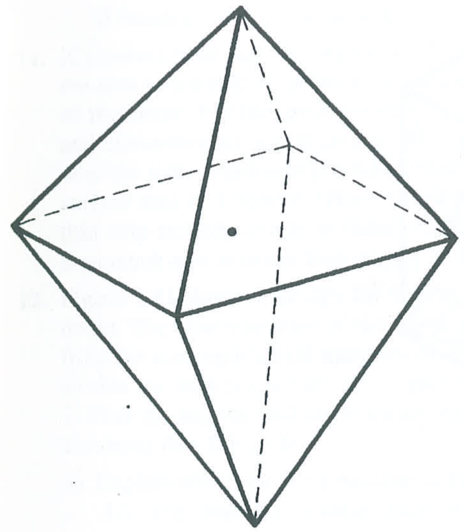

Figure 1.56

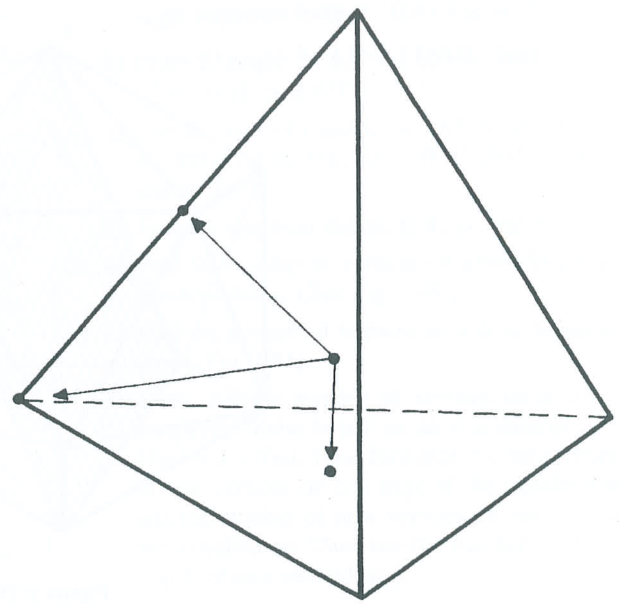

Figure 1.57

Theorem 1.6.3, show that excesses are additive: If D is on the short arc of the great circle through B and C, the excess of $\triangle ABC$ is the sum of the excesses of $\triangle ABD$ and $\triangle ADC$. Illustrate this situation.

9. On a sphere let N be the "north pole" and A and B be two points on the "equator" with $m\angle ANB = 90°$. Let C, D, and E be the midpoints of $\widehat{AB}$, $\widehat{AN}$, and $\widehat{BN}$, respectively.

a) Explain and illustrate why $\widehat{CN}$, $\widehat{DB}$, and $\widehat{AE}$ intersect in a common point, F.

b) Find the angle sum of the spherical triangle $\triangle ACF$.

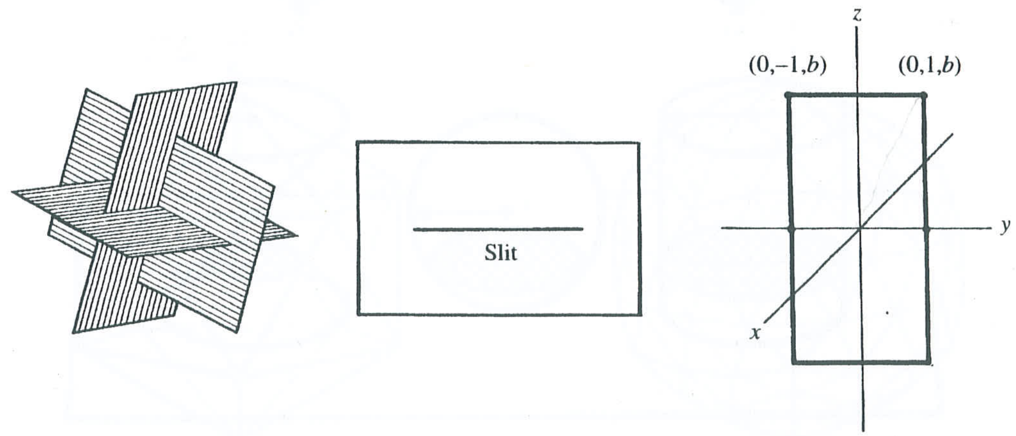

Figure 1.58

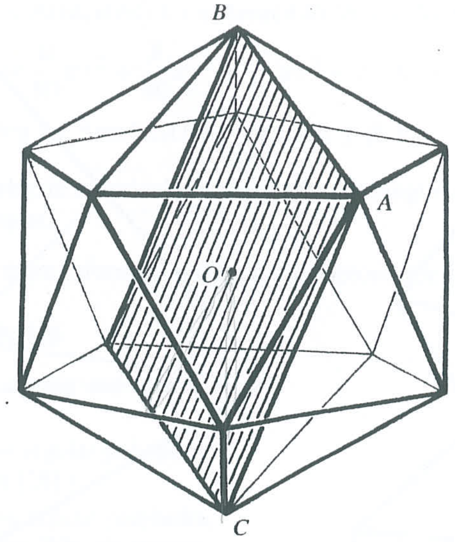

Figure 1.59

c) Verify that the area of $\triangle ACF$ agrees with Theorem 1.6.3.

10. Cavalieri's principle is often an axiom in high school texts because it provides an elementary way to prove results about volumes of curved shapes. Bonaventura Cavalieri (1598–1647), a pupil of Galileo, used his principle to find volumes (and areas) before the advent of calculus.

Cavalieri's Principle: Let A and B be two solids included between two parallel planes. If every plane P parallel to the given planes intersects A and B in sections with the same area, then A and B have the same volume.

a) Use Cavalieri's principle and Fig. 1.60 to show the following result of Archimedes. The volume of a cylinder whose height is twice its radius equals the volume of a sphere of the same radius plus twice the volume of a cone with the same radius and a height equal to the radius.

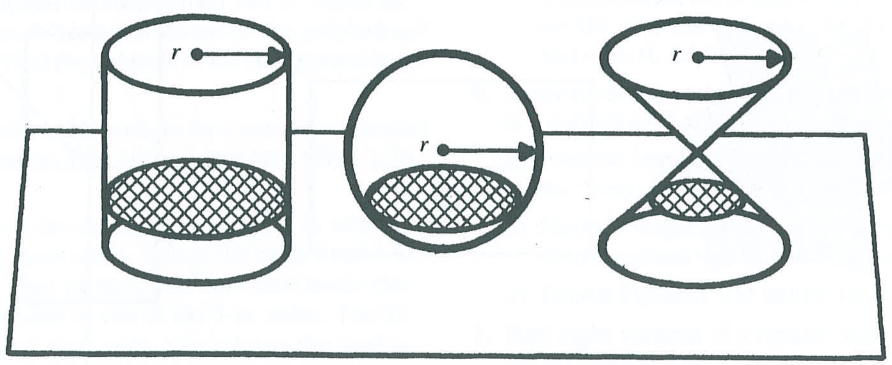

Figure 1.60 Compare the areas of the cross sections of a cylinder, a sphere, and a double cone at the same height.

The family of planes must together completely cover both solids.

b) Find two solids A and B with different volumes and a family of nonparallel planes so that A and B intersect each plane in sections of equal area.

11. (Calculus) Note that the derivative dA/dr of πr^2, the area of a circle of radius r, is the circumference of the circle. Explain geometrically why the area and circumference are related in this way. Explain why the same relationship holds for the volume and surface area of a sphere. [*Hint:* Consider adding a thin strip around a circle of radius r. Approximately how much area is in the thin strip?]

12. Figure 1.61 shows a design for a three-frequency dome. The nine triangles of the dome are "lifted" from the nine equilateral triangles dividing $\triangle ABC$ so that the vertices are all on a sphere of radius 1. Find the lengths and angle measures as follows, assuming that $DE = DI$.

a) Explain why there are just three different lengths, AD, DE, and DJ. Explain how to find all the angle measures from $m\angle DAI$ and $m\angle DJE$.

b) From Example 3, $AB \approx 1.05146$. Find $D'E'$, $D'M'$, OM', and OD'.

c) Use the law of cosines to find $m\angle D'OE'$, $m\angle AOD'$, AD, DE, OJ', $D'J'$, DJ, $m\angle DAI$, and $m\angle DJE$.

d) Find the spherical excess at A, D, and J.

13. a) Find the number of vertices on a two-frequency geodesic dome. (See Fig. 1.49.)

b) Find the number of vertices on a three-frequency dome (Fig. 1.61).

c) Show that the number of vertices on an n-frequency dome based on an icosahedron is $10n^2 + 2$. [*Hint:* Find formulas for the number of new vertices on one edge of the icosahedron and the number of new vertices on one face of the icosahedron. Then use the numbers of V, E, and F of an icosahedron.]

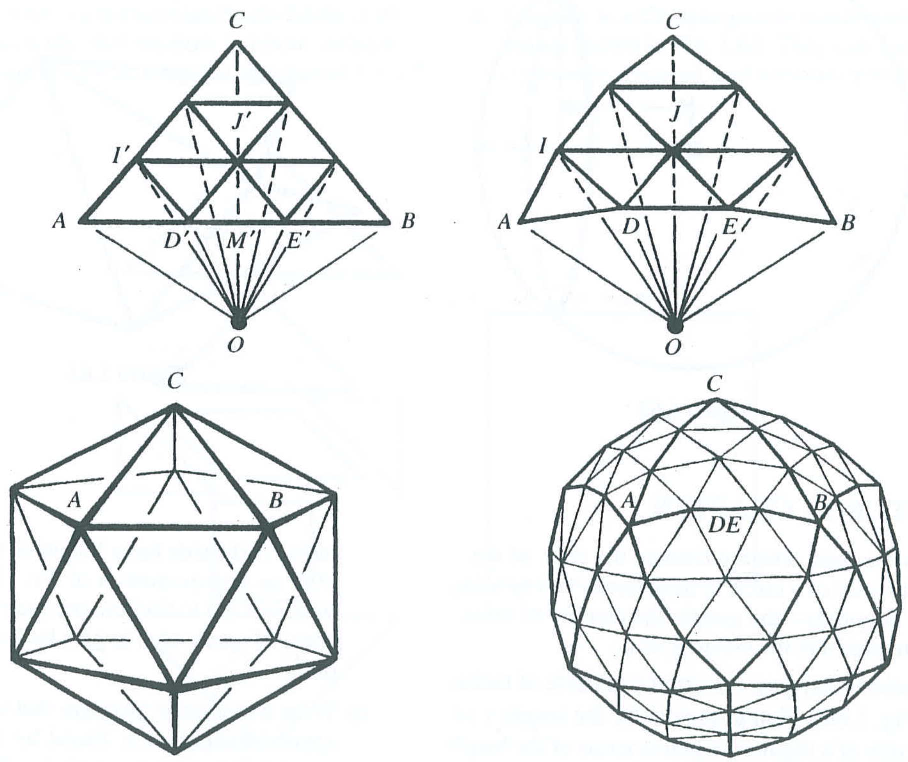

Figure 1.61 A three-frequency dome.

d) Use part (c) to find the average spherical excess at each vertex of an n-frequency dome. Compare your value when $n = 3$ with your answer in Problem 12(d). What is the largest practical value of n if the average spherical excess must be $\frac{1}{2}^\circ$ for a dome?

14. Give an example on a sphere to show that Euclid I-16 is independent of Euclid's first four postulates.

15. A *spherical cap* is the set of all points on a sphere of radius R whose distance (measured along the sphere) is at most r from a point A on the sphere.

a) Find the circumference of the spherical cap in terms of r and R.

b) (Calculus) Verify that the area of the spherical cap is $2\pi R^2 - 2\pi R^2 \cos(r/R)$. [*Hint:* Use radians. Let x be as shown in Fig. 1.62. Recall that the area of the surface obtained by revolving $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is $\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}dx$. Calculate the integral for a spherical cap for general a and b. Then determine a and b in terms of r and R.]

c) Archimedes found the surface area of the spherical cap to equal the area of a circle whose radius equals the (straight-line) distance from A to a point on the circumference of the cap. Verify Archimedes' theorem.

16. Curiously, the Euclidean plane is "triangle complete" but Euclidean space is not "tetrahedron complete." Triangle complete means that any three lengths that satisfy the triangle inequality ($a + b \geq c$) in any order can actually appear as the sides of a Euclidean triangle. However, there are sets of six lengths that satisfy the triangle inequality appropriately, but no tetrahedron in Euclidean space has its six edges with those lengths. More easily, there are four triangles whose corresponding sides are congruent that cannot be folded to make a tetrahedron. Find such an example. (Figure 1.63 shows four triangles that can be folded to make a tetrahedron.)

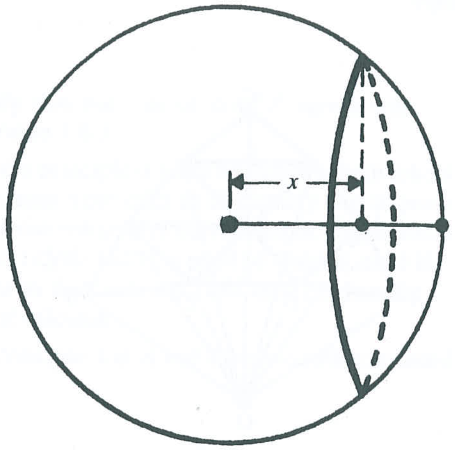

Figure 1.62

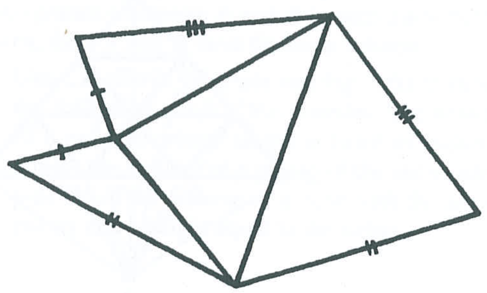

Figure 1.63

PROJECTS FOR CHAPTER 1

1. Archimedes and others estimated the ratio of the circumference of a circle to its diameter (π) by using regular polygons—the greater the number of sides, the more accurate the estimate of π.

a) Consider polygons inscribed in a circle of radius 1 (Fig. 1.64). Find a formula for the length y of the side of a regular $2n$-gon in terms of the length x of the side of a regular n-gon.

b) For a regular hexagon inscribed in a circle of radius 1, the side has a length of 1 and a perimeter of 6, an approximation of 2π. Use the formula from part (a) to find the perimeters of regular 12-gons, 24-gons, etc., to give better lower estimates of 2π.

c) Write a computer program that will print out the approximations of π found by the formula in part (a) for polygons with 3×2^i sides, where $1 \leq i \leq 30$.

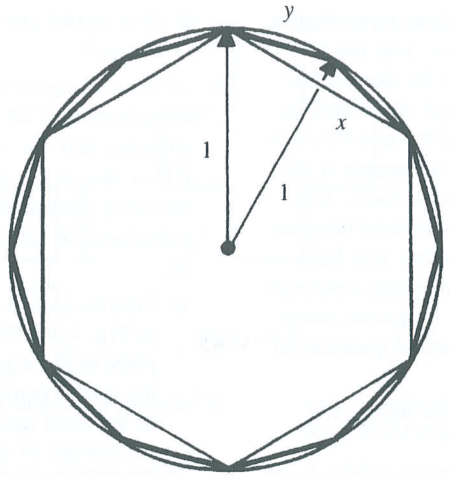

Figure 1.64 Find y in terms of x.

d) Find upper estimates for π by using circumscribed regular polygons in parts (a), (b), and (c). [*Hint:* The formula for $\tan(2\theta)$ will help.]

Remark The Arab mathematician Nasir-Eddin (1201–1274) used essentially this method (without modern notation or computers!) with a regular polygon of 3×2^{28} sides. He wanted to ensure that the error in his estimate of the circumference of the universe, based on Islamic ideas of its radius, was no more than the width of a horse's hair.

2. Tangram is a Chinese puzzle made with the seven shapes shown in Fig. 1.65. They can form a square, if properly arranged, and various interesting shapes.

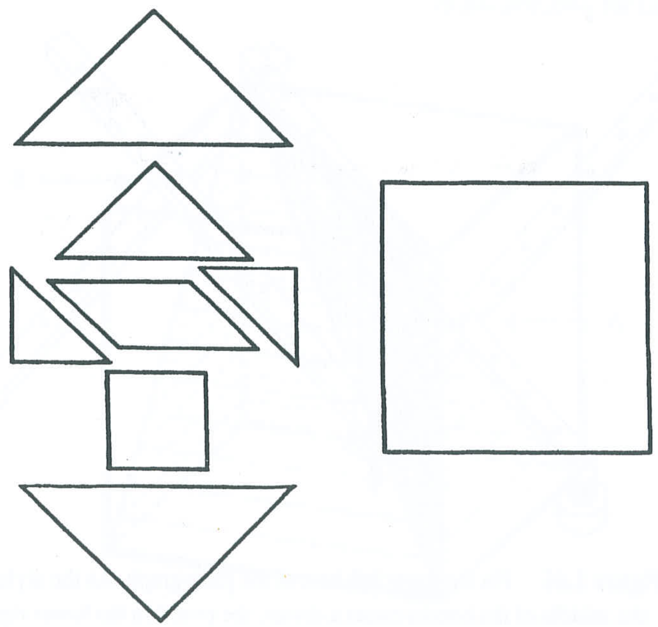

Figure 1.65 The seven pieces of a tangram can be arranged to form a square.

Any figure that can be made with these seven shapes is called a *tangram*. In this project, you are to construct all convex tangrams. Let the short sides of the smallest triangle have a length of 1. Find the lengths of the sides, the angles, and the area for each of the seven shapes. What angles can appear at the corners of a convex tangram? Why is there only one triangular tangram? Why can a convex tangram not have more than eight sides? After you have found the candidates for convex tangrams, construct them. (Read [24] gives all convex tangrams, some background, and an open mathematical question on tangrams.)

3. Figure 1.66 shows a pantograph, a device for enlarging a design.

a) Explain why the design traced by the pencil must always be similar to the original design. [*Hint:* Why do the bars need to form a parallelogram?]

b) Determine where the holes should be drilled so that the pencil will trace a figure whose dimensions are twice the original design and k times the original design.

c) Make a pantograph and use it to enlarge several designs.

d) How could you alter a pantograph to reduce a design?

4. Let's try to measure how close a nonconvex set is to being convex. Call a point P in a set a *guard point* provided that for every other point Q in the set, $\overline{PQ}$ is also in the set. (A guard at a guard point can see every spot in the set.) The ratio (area of guard points/area of set) is called the *inside convexity* of a set.

a) Find the inside convexity for each shape depicted in Fig. 1.67. For each shape, find the "worst" place to put a guard, that is, the point from which the guard would see the smallest percentage of the total area. From that worst place, what percentage of the entire area does a guard see? Explain your answers.

Another way to measure the convexity of a set is to measure how much must be added to the set to obtain a convex set. Define the *outside convexity* of a set S to be the ratio (area of S/area of the smallest convex set containing S).

b) Find the outside convexity of the shapes shown in Fig. 1.67.

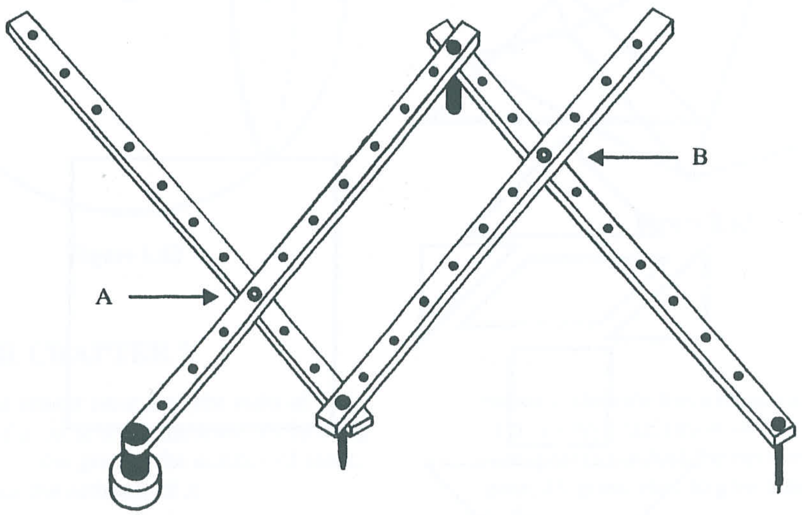

Figure 1.66 Fix the lower left base of the pantograph. As the stylus in the middle of the bottom traces a design, the pencil in the lower right draws an enlargement. Moving the connectors at A and B change the scale of the enlargement.

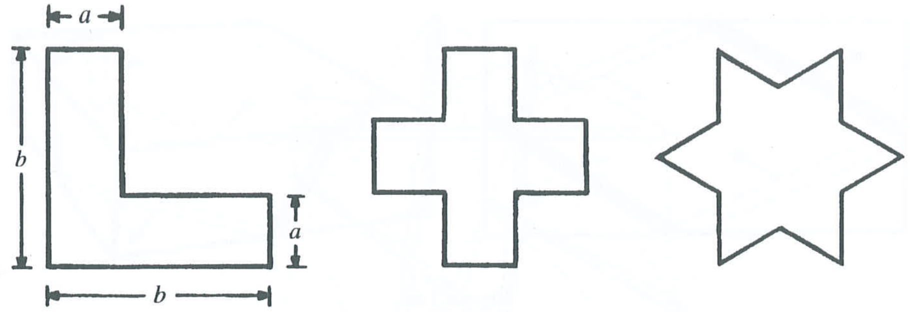

Figure 1.67

c) Explain why the inside (or outside) convexity of any convex set is 1.

d) Find a set with positive area whose inside (outside) convexity is 0.

e) Find a nonconvex set whose inside and outside convexity is 1. Explain.

f) Compare the definitions of *inside* and *outside convexity* with your intuition of how close a set is to convex. Look for a better definition.

5. Use Fig. 1.68 to give a plausible explanation of why the volume of a square pyramid is one-third the height times the base. Build a model of Fig. 1.68. Will this decomposition work if the cube is replaced with a rectangular box? Will this decomposition work if the cube is replaced with a triangular prism?

6. A ribbon wrapped around a box can be removed without cutting, stretching, or untying it (Fig. 1.69). Try to do so with an actual ribbon on a box. Then model this situation geometrically and explain why it works. It may be easier to explain first with a rectangle.

7. a) Construct the 13 Archimedean solids (Wenninger [28]). The Archimedean solids, together with an infinite family of prisms and another infinite family of antiprisms, are semiregular polyhedra. *Semiregular polyhedra* have two or more kinds of regular polygons for faces and the same number,

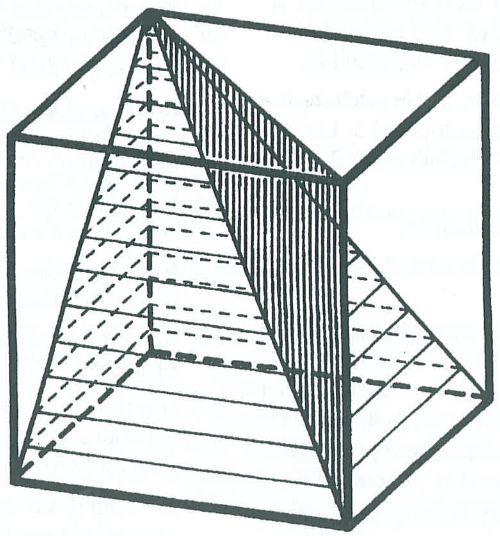

Figure 1.68

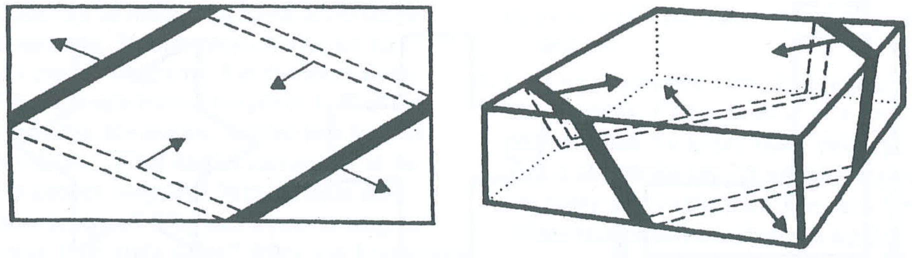

Figure 1.69

types, and arrangement of faces around each vertex.

b) Find the spherical excess at a vertex for each of these Archimedean solids.

c) Find the spherical excess at a vertex for semiregular prisms and antiprisms.

d) Explain why there can be no other Archimedean solids.

8. Define a *deltahedron* to be a strictly convex polyhedron all of whose faces are equilateral triangles. (A convex polyhedron is strictly convex if no two faces lie in the same plane. We use the name deltahedron because the capital Greek letter delta Δ looks like a triangle.)

a) Find an equation relating E and F in deltahedra. Justify your equation.

b) Explain why there are at most five triangles at any vertex of a deltahedron. Use this to find an inequality relating V and E in a deltahedron.

c) Use parts (a) and (b) to show that in a deltahedron $E \leq 30$ and E must be a multiple of 3. List all the possible candidates for values of V, E, and F of deltahedra.

d) Build a complete set of deltahedra.

Remarks There is no deltahedron with $E = 27$. (See Beck et al. [2].)

9. Euler's formula applies to various polyhedra that are not convex. Imagine polyhedra as made of rubber and inflate them. Then the ones that can be inflated to look like a sphere will satisfy Euler's formula. The generalized Euler's formula applies to polyhedra with holes. Let H be the number of holes in the polyhedron. Find an equality relating V, E, F, and H. (The value $H = 0$ should give you Euler's formula.) You will probably need to draw or make several examples. You may find that you need to make the definition of a polyhedron explicit. Beck et al. [2] gives applications of this generalized formula. Generalize Descartes's formula to polyhedra with holes.

10. For a convex polyhedron, form its *dual* polyhedron as follows. The vertices of the dual are at the centers of the original polyhedron's faces. Two new vertices are connected with an edge provided that the corresponding original faces were joined at an edge.

a) Describe the faces of the dual.

b) Verify that the duals of regular polyhedra are again regular polyhedra.

c) How are V, E, and F for a polyhedron and its dual related?

d) Build duals of some of the Archimedean solids.

11. Build a geodesic dome. (See Kenner [17].)

12. Build some tensegrity figures, invented by Buckminster Fuller, such as that shown in Fig. 1.70. (See Kenner [17] and Pugh [23].)

13. Investigate the Geometer's Sketchpad, CABRI, or other computer programs designed to aid geometric exploration.

14. Investigate the geometry of four and more dimensions. (See Coxeter [5] and Rucker [25].)

15. Investigate the history of geometry. (See Aaboe [1], Eves [9], Kline [18], and Struik [27].)

16. Investigate axiomatic systems. (See Bryant [4] and Fishback [10].)

17. Develop an axiomatic system describing the geometry of pixels (points) and lines on a computer screen. Find and prove some theorems in this geometry.

18. Investigate the golden ratio and phyllotaxis. (See Coxeter [5] and Huntley [16].)

19. Investigate taxicab geometry. (See Krause [19].)

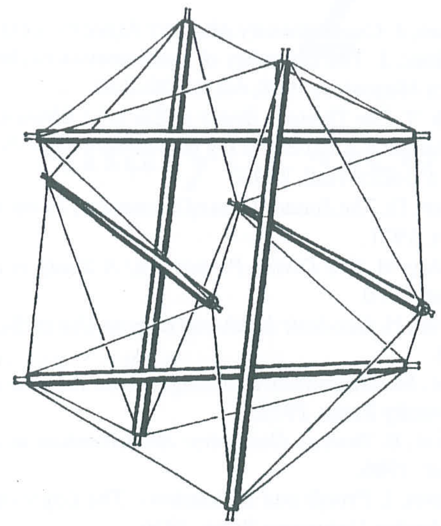

Figure 1.70

20. Investigate the geometry of the sphere. (See Henderson [14] and McCleary [21].)

21. Write an essay giving your understanding of why mathematics is certain and why it is applicable. Compare your ideas with those of Plato and Aristotle. (See also Grabiner [12].)

22. Write an essay on the roles of intuition and proofs in geometric understanding.

23. Write an essay on the different levels of proof demanded in mathematics, in a civil court case ("the preponderance of the evidence") and a criminal court case ("beyond a reasonable doubt"). What phrase would you offer to describe mathematical proofs? Compare the advantages and disadvantages of these differing levels of proof.

24. Write an essay explaining axiomatic systems and models to a high school geometry student. Use examples, preferably everyday ones.

25. Write an essay on the understanding of definition in mathematics, in other disciplines and in every day language. Discuss the advantages and disadvantages of these different notions of definition.

Suggested Readings

[1] Aaboe, A. *Episodes from the Early History of Mathematics*. Washington, D.C.: Mathematical Association of America, 1964.
[2] Beck, A., M. Bleicher, and D. Crowe. *Excursions into Mathematics*. New York: Worth, 1969.
[3] Boltyanskii, V. *Equivalent and Equidecomposable Figures*. Lexington, Mass.: D. C. Heath, 1963.
[4] Bryant, V. A fresh look at geometry. *Mathematics Magazine*, 1971, 44(1):180–182.
[5] Coxeter, H. *Introduction to Geometry*. New York: John Wiley & Sons, 1969.
[6] DeLong, H. *A Profile of Mathematical Logic*. Reading, Mass.: Addison-Wesley, 1971.
[7] Dubnov, I. *Mistakes in Geometric Proofs*. Lexington, Mass.: D. C. Heath, 1963.
[8] Edmondson, A. *A Fuller Explanation: The Synergetic Geometry of R. Buckminster Fuller*. New York: Birkhäuser, 1987.
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[10] Fishback, W., *Projective and Euclidean Geometry*. New York: John Wiley & Sons, 1962.

[11] Gallian, J. *Contemporary Abstract Algebra*. Lexington, Mass.: D. C. Heath, 1990.
[12] Grabiner, J. The centrality of mathematics in the history of Western thought, *Mathematics Magazine*, 1988, 61(4):220–230.
[13] Heath, T. *The Thirteen Books of Euclid's Elements*, 3 vols. Mineola, NY: Dover, 1956.
[14] Henderson, D. *Experiencing Geometry on the Plane and Sphere*. Upper Saddle River, N.J.: Prentice-Hall, 1996.
[15] Hilbert, D. *The Foundations of Geometry*, 2d ed. (transl. E. Townsend). Peru, Ill.: Open Court, 1921.
[16] Huntley, H. *The Divine Proportion: A Study in Mathematical Beauty*. Mineola, N.Y.: Dover, 1970.
[17] Kenner, H. *Geodesic Math and How to Use It*. Berkeley: University of California Press, 1976.
[18] Kline, M. *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press, 1972.
[19] Krause, E. *Taxicab Geometry: An Adventure in Non-Euclidean Geometry*. New York: Dover, 1986.
[20] Lakatos, I. *Proofs and Refutations: The Logic of Mathematical Discovery.* New York: Cambridge University Press, 1976.
[21] McCleary, J. *Geometry from a Differentiable Viewpoint*. Port Chester, N.Y.: Cambridge University Press, 1994.
[22] McMahon, T., and J. Bonner. *On Size and Life.* New York: Scientific American Library, Scientific American Books, 1983.
[23] Pugh, A. *An Introduction to Tensegrity*. Berkeley: University of California Press, 1976.
[24] Read, R. *Tangrams*. Mineola, N.Y.: Dover, 1965.
[25] Rucker, R. *Geometry, Relativity and the Fourth Dimension*. Mineola, N.Y.: Dover, 1977.
[26] Salmon, W. *Zeno's Paradoxes*. Indianapolis: Bobbs-Merrill, 1970.
[27] Struik, D. *A Concise History of Mathematics*. Mineola, N.Y.: Dover, 1967.
[28] Wenninger, M. *Polyhedron Models for the Classroom*. Reston, Va.: National Council of Teachers of Mathematics, 1966.
[29] Wilder, R. *The Foundations of Mathematics*. New York: John Wiley & Sons, 1965.

Suggested Media

1. "Curves of Constant Width," 16-minute film, International Film Bureau, Chicago, Ill., 1971.
2. "Equidecomposable Polygons," 10.5-minute film, International Film Bureau, Chicago, Ill., 1971.
3. "Geodesic Domes: Math Raises the Roof," 20-minute film, David Nulsen Enterprises, Santa Monica, Calif., 1979.
4. "Geometry: New Tools for New Technologies," 60-minute video, COMAP, Lexington, Mass., 1993.
5. "Mathematics of the Honeycomb," 13-minute film, Moody Institute of Science, Whittier, Calif., 1964.
6. "Music of the Spheres," 52-minute video from *The Ascent of Man* series, Time–Life Multimedia, New York, 1974.
7. "The Theorem of Pythagoras," 26-minute video, California Institute of Technology, Pasadena, Calif., 1988.
8. "The Tunnel of Samos," 30-minute video, California Institute of Technology, Pasadena, Calif., 1995.