1.5 Similar Figures

The mathematical aspects of scale changes have been studied for thousands of years but continue to yield insights in various fields. The Euclidean tradition focuses on similar triangles and polygons. In Chapter 4 we use transformations to provide a deeper treatment of similar figures.

Definition 1.5.1 Nonzero numbers $a_1, a_2, \ldots$ are *proportional* to $b_1, b_2, \ldots$ iff there is a nonzero number k such that $k \cdot a_i = b_i$. The number k is the *ratio of proportionality*.

Exercise 1 Show that a_1, a_2 are proportional to b_1, b_2 only if $a_1 \cdot b_2 = a_2 \cdot b_1$.

Definition 1.5.2 Triangles $\triangle ABC$ and $\triangle A'B'C'$ are *similar* iff corresponding angles are congruent and the lengths of corresponding sides are proportional. We write $\triangle ABC \sim \triangle A'B'C'$.

Example 1 Show that the three triangles in Fig. 1.37 are similar.

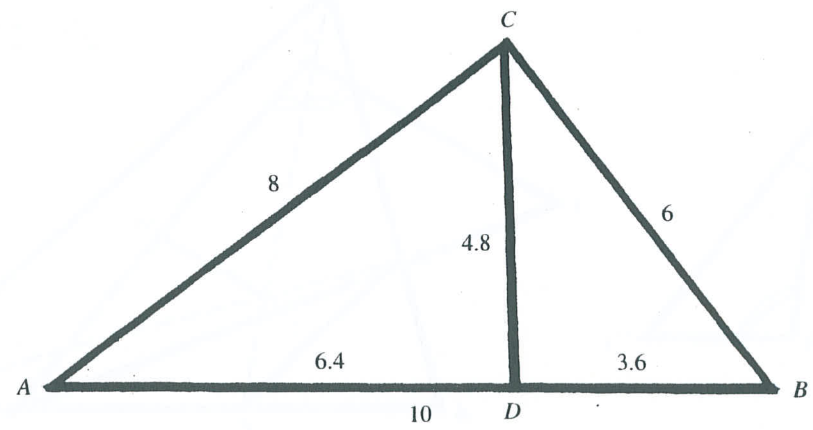

Figure 1.37

Solution. Each triangle is a right triangle by the converse of the Pythagorean theorem (Euclid I-48). Elementary trigonometry confirms that the other angles are congruent. Furthermore, the sides of the largest triangle (10, 6, and 8) and the smallest triangle (6, 3.6, and 4.8) are proportional, using $k = 0.6$. Similarly, the ratio $k = 0.8$ converts the sides of the largest triangle to the sides of the middle triangle (8, 4.8, and 6.4). ●

To show two triangles to be similar, we must in principle prove the congruence or proportionality for all their corresponding parts. Similarity theorems provide smaller sufficient sets of corresponding parts, analogous to the congruence theorems SAS, AAS, and SSS. We follow Euclid's approach, supplemented with our modern understanding of real numbers, to develop these theorems. Theorem 1.5.1 is Euclid's key to proving Theorems 1.5.2, 1.5.3, and 1.5.4.

Theorem 1.5.1 (Euclid VI-2) Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be two rays with D between A and B and E between A and C. Then $\overline{BC} \| \overline{DE}$ iff $\triangle ABC \sim \triangle ADE$.

Proof. Construct segments $\overline{BE}$ and $\overline{CD}$, as in Fig. 1.38. First suppose that $\overline{BC} \| \overline{DE}$. Then the corresponding angles are equal by I-29. Triangles $\triangle BDE$ and $\triangle CED$ have the same area because they have the same base ($\overline{DE}$) and the same height (the distance between the parallel lines). Thus the areas of these triangles have the same ratio to the area of $\triangle ADE$, which also has base $\overline{DE}$. Now reconsider $\triangle ADE$ and $\triangle DBE$, using $\overline{AD}$ and $\overline{DB}$ as the bases so that these two triangles have the same height. Hence the lengths of their bases and their areas are proportional because of the formula Area = $\frac{1}{2}$(Base)(Height). That is, $\frac{1}{2}$(Height) is the ratio of proportionality. The same can be said about the triangles $\triangle ADE$ and $\triangle DCE$. Thus the bases $\overline{AD}$ and $\overline{DB}$ are in proportion to $\overline{AE}$ and $\overline{EC}$. By the addition properties of proportions, this result shows $\overline{AD}$ and $\overline{AB}$ are proportional to $\overline{AE}$ and $\overline{AC}$.

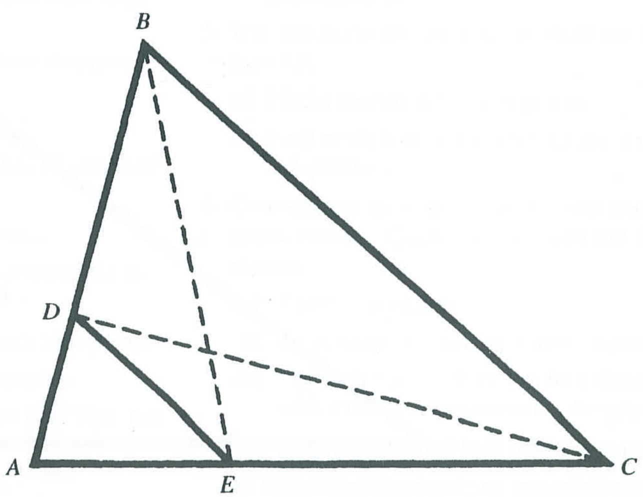

Figure 1.38

Problem 7 asks you to prove that the sides $\overline{DE}$ and $\overline{BC}$ have the same proportion as the other two pairs of sides. Problem 8 asks you to show the other direction. ■

Theorem 1.5.2 (Euclid VI-4) If two triangles have two pairs of corresponding angles congruent, then the triangles are similar.

Proof. Let $\triangle ABC$ and $\triangle A'B'C'$ have $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$. By Theorem 1.1.1, $\angle C \cong \angle C'$. We must show that the three pairs of sides are proportional. If $\overline{AB} \cong \overline{A'B'}$, then the triangles are congruent by AAS. So, WLOG, suppose that $\overline{AB}$ is longer than $\overline{A'B'}$. Construct D on $\overline{AB}$ such that $\overline{AD} \cong \overline{A'B'}$, and construct $\overline{DE} \| \overline{BC}$ with E on $\overline{AC}$, as in Fig. 1.39. Then $\triangle ADE \cong \triangle A'B'C'$ by AAS. Furthermore, $\triangle ABC \sim \triangle ADE$ by Theorem 1.5.1. Then, by the definitions of congruent and similar triangles, $\triangle ABC \sim \triangle A'B'C'$. ■

Theorem 1.5.3 (Euclid VI-5) If the three pairs of corresponding sides of two triangles are in proportion, then the two triangles are similar.

Proof. Problem 9. ■

Theorem 1.5.4 (Euclid VI-6) If two pairs of corresponding sides of two triangles are proportional and their included angles are congruent, then the triangles are similar.

Proof. Problem 10. ■

Definition 1.5.3 The *area* of a rectangle is the product of the lengths of two of its adjacent sides.

Exercise 2 Use Fig. 1.40 to derive the formula for the area of a triangle.

Theorem 1.5.5 If the lengths of the sides of $\triangle ABC$ are k times the lengths of the corresponding sides of $\triangle A'B'C'$, then the area of $\triangle ABC$ is k^2 times the area of $\triangle A'B'C'$.

Proof. Problem 11. ■

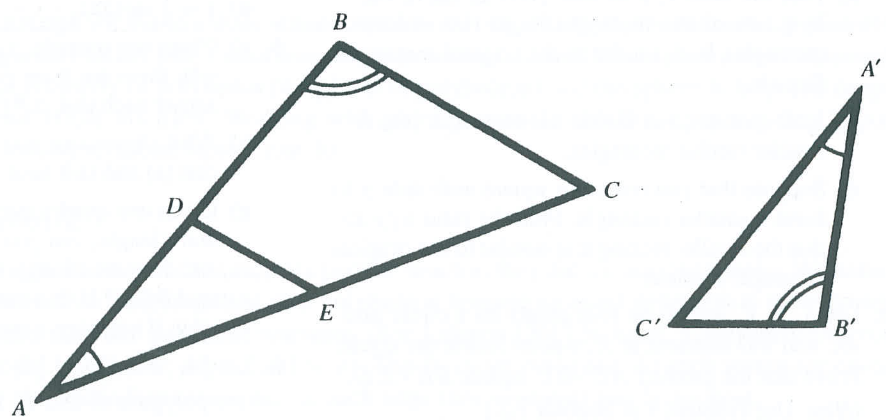

Figure 1.39

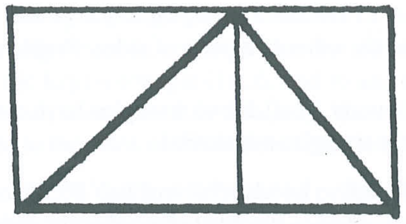

Figure 1.40

The generalization of Theorem 1.5.5 for areas and volumes of similar figures has important consequences for animals of different sizes. To state the generalization, let's suppose that there is a scaling ratio of k from figure A to B. Then the area (or surface area) of B is k^2 times the area of A, and the volume of B is k^3 times the volume of A. Consider increasing the size of an animal. The strength of bones and muscles grows roughly with the areas of their cross sections (k^2). However, the weight these bones and muscles must handle grows roughly with the volume of the animal (k^3). As animals get larger, the bones and muscles must get disproportionately thicker to support and move the greatly increased weight. Similarly, the lift of a bird's wings is roughly proportional to the surface area of the wings. The bird's weight is roughly proportional to the volume of the bird. To compensate for their much greater weights, large birds have evolved greater wingspreads than small birds. (McMahon and Bonner [22] provide more information on area and volume questions in nature.)

PROBLEMS FOR SECTION 1.5

1. Explain why in any triangle $\triangle ABC$ there is a unique point D on $\overrightarrow{AC}$ such that $\triangle ABC \sim \triangle ADB$. Describe AD in terms of the sides of $\triangle ABC$.

2. Consider a rectangle with length x and width y, where $x > y$.

a) Find the ratio x/y so that you can halve the long side of this rectangle to get two smaller rectangles, both similar to the original rectangle. Explain.

b) Redo part (a), but divide the rectangle into n smaller similar rectangles.

c) Suppose that you remove a square with side y to leave a smaller rectangle. Find the ratio x/y so that the smaller rectangle is similar to the original rectangle. Explain.

3. Let A, B, C, and D be four points on a circle and $\overline{AC}$ and $\overline{BD}$ intersect at E, a point inside the circle. Prove that the product $AE \cdot EC$ equals $BE \cdot ED$. [*Hint:* Use Problem 9 of Section 1.2.]

4. Which of the following functions have graphs that are similar to the graph of $y = \sin(x)$? Explain your answer by using their graphs and your intuitive understanding of similarity. Generalize this problem.

a) $y = 2\sin(x)$

b) $y = \sin(2x)$

c) $y = 2\sin(2x)$

d) $y = \frac{1}{2}\sin(2x)$

5. a) Given any triangle $\triangle ABC$ and a square, explain why there are three points P, Q, and R on the square such that $\triangle PQR \sim \triangle ABC$.

b) What shapes can you substitute for the square in part (a) and still have the result hold? Explain.

c) Given any quadrilateral $ABCD$ and an equilateral triangle, can you find four points P, Q, R, and S on the triangle such that $PQRS$ is similar to $ABCD$? If this can work in general, explain why; if not, give a counterexample.

6. Let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be the vertices of two polygons A and B, respectively. Then define polygons A and B to be *similar* ($A \sim B$) iff both

i) there is a positive r such that for all $i, j \in I$, $A_iA_j = rB_iB_j$ *and*

ii) for all $i, j, k \in I$, $\angle A_iA_jA_k \cong \angle B_iB_jB_k$.

a) How does this definition compare with your intuition of similar polygons?

b) Prove that any two squares are similar.

c) Provide an example of two quadrilaterals whose sides are proportional but are not similar.

d) Provide an example of two quadrilaterals whose angles are congruent but are not similar.

e) Prove that any two regular polygons with the same number of sides are similar.

f) Prove that $A \sim B$ provided that for all $i, j, k \in I$, $\triangle A_iA_jA_k \sim \triangle B_iB_jB_k$.

g) If you allow the index set I to be infinite, does the given definition of *similar* still make sense? If so, give an example; if not, explain why not.

7. Fill in the proof of Theorem 1.5.1 by showing that the sides $\overline{DE}$ and $\overline{BC}$ are in the same proportion as the other two pairs of sides. [*Hint:* Construct $\overline{EF} \parallel \overline{AB}$ with F on $\overline{BC}$.]

8. Prove the remaining direction of Theorem 1.5.1.

9. Prove Theorem 1.5.3.

10. Prove Theorem 1.5.4.

11. Prove Theorem 1.5.5.

12. Generalize Theorem 1.5.5 to polygons with n sides, using induction. Assume that any polygon can be divided into triangles.

13. (Calculus) Recall that the area bounded by $y = f(x)$, the x-axis, $x = a$, and $x = b$ is $\int_a^b f(x)dx$ (Fig. 1.41).

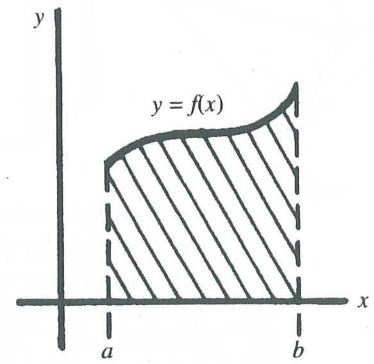

Figure 1.41

a) Explain why the region bounded by $y = kf(x/k)$, the x-axis, $x = ka$, and $x = kb$ is similar to the region shown in Fig. 1.41. In particular, explain why you need to use $f(x/k)$ instead of $f(x)$.

b) Use calculus to verify that the area of the region in part (a) is k^2 times the area of the region shown in Fig. 1.41.

c) Generalize this problem to volumes of similar regions in three dimensions.