1.4 Axiomatic Systems, Models, and Metamathematics

In Section 1.3 we showed the logical need for undefined terms in an axiomatic system, which ignores what those terms mean. However, people depend on meaning and intuition to create and understand mathematics. A string stretched taut between its ends provides a strong intuition of a line, but such an image is too imprecise for mathematics. Mathematical models provide an explicit link between intuitions and undefined terms. The usual analytic model of Euclidean plane geometry is the set $\mathbf{R}^2 = \{(x, y) : x, y \in \mathbf{R}\}$, where a *point* is interpreted as an ordered pair of numbers (x, y) and a *line* is interpreted as the points that satisfy an appropriate first degree equation $ax + by + c = 0$. High school students spend considerable time learning how this algebraic model matches geometric intuition and axioms. In making a model, we are free to interpret the undefined terms in any way we want, provided that all the axioms hold under our interpretation. Note that the axioms are not by themselves true; a context is needed to give meaning to the axioms in order for them to be true or false.

Definition 1.4.1 A *model* of an axiomatic system is a set of objects together with interpretations of all the undefined terms of the axiomatic system such that all the axioms are true in the set using the interpretations.

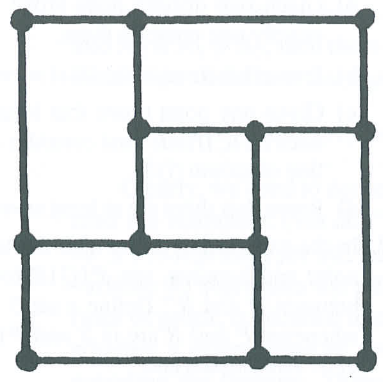

Figure 1.29 Interpret *point* by a dot and *line* by a line segment.

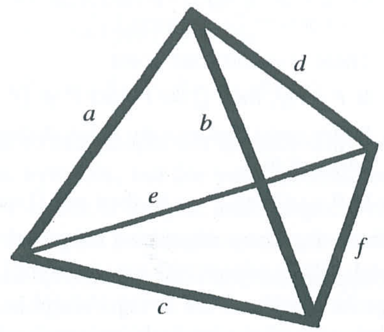

Figure 1.30 Interpret *point* by a line segment and *line* by a face of the pyramid.

Example 1 Figures 1.29 and 1.30 give different models of the axiomatic system with the undefined terms *point*, *line*, and *on* and the axioms:

i) Every point is on exactly two lines.
ii) Every line is on exactly three points.

In these and later models *on* has the obvious interpretation. ●

Exercise 1 In the model given by Fig. 1.30 reinterpret *lines* as the vertices (corners) of the pyramid (and *points* as line segments.) Is the result still a model of the axioms in Example 1.4.1?

Models do much more than provide concrete examples of axiomatic systems; they lead to important understandings about axiomatic systems. *Metamathematics* considers axiomatic systems and their models as a whole. The prefix *meta*, Greek for "beyond," distinguishes metamathematics from the mathematical properties proved within a system. For example, Metatheorem 1 implies that the axioms of a system are all that are needed to determine the models of the system. The proofs of metatheorems are beyond the level of this book. (See DeLong [6, Chapter 4] and Wilder [29, Chapter 2] for more on metamathematics.)

Metatheorem 1.4.1 If all the axioms of an axiomatic system are true in a model, then all the theorems of that system are also true in that model.

The most important property of an axiomatic system is consistency, which says that we cannot prove two statements that contradict each other. For example, a proof of both $1 = 2$ and $1 \neq 2$ would render proofs useless. However, consistency is difficult and often impossible to show directly. Fortunately Metatheorem 1.4.2 provides an easier method with models. In particular, the models for Example 1 show that the axiomatic system is consistent.

Definition 1.4.2 An axiomatic system is *consistent* iff no contradictions can be proven from the axioms. An axiomatic system is *relatively consistent* iff its consistency can be proven assuming the consistency of another axiomatic system.

Metatheorem 1.4.2 (Gödel, 1930) An axiomatic system is consistent iff it has a model.

The analytic model of geometry shows the relative consistency of Euclidean geometry because analytic geometry depends on the real number system. Are the axioms for the real numbers consistent? The model of the real numbers we are likely to choose is a geometric one: a line in Euclidean geometry. We risk entering a vicious circle: The crucial property of a mathematical system, consistency, is beyond our reach for two central mathematical systems. Indeed, this problem and others like it inspired David Hilbert and others to investigate mathematical logic. Mathematical logic has given us profound insights into the nature of mathematics and proof, including their limitations. In particular, one of Gödel's incompleteness theorems, proven in 1930, shows that we cannot prove the absolute consistency of elementary arithmetic, and by extension, Euclidean geometry and the real number system. Thus relative consistency is the best we can do for such sophisticated systems. No one seriously doubts the consistency of these systems.

Other metamathematical properties, though not essential, give important insight into axiomatic systems and models.

Definition 1.4.3 A statement is *independent* of a set of axioms iff neither the statement nor its negation can be proved from the axioms. If, in an axiomatic system, each axiom is independent of the other axioms, the set of axioms is said to be *independent*. An axiomatic system is *complete* iff every statement based on the undefined terms can either be proved or disproved from the axioms.

We can use Metatheorem 1.4.1 to prove independence with models. Suppose that we have two models for a set of axioms and that a statement is true in the first model but not in the second model. Then Metatheorem 1.4.1 applied to the second model shows that the statement cannot be a theorem of these axioms. Similarly, the first model shows the negation of that statement cannot be a theorem.

Example 2 The models in Figs. 1.29 and 1.30 show the statement "there are exactly six points" to be independent of the axioms of Example 1. Because this statement is independent of the axioms, these axioms are not complete. ●

Example 3 Taxicab geometry alters the usual analytic model of Euclidean geometry by giving a different interpretation to distance. Rather than use the Pythagorean theorem to measure distance, we use the formula $d_T((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$. This formula gives the distance a taxi goes if it travels only along north–south and east–west streets (Fig. 1.31). This model still satisfies Euclid's postulates, although circles look odd. However, Euclid I-4 (SAS) is no longer valid in this geometry (Fig. 1.32). This theorem holds in the usual analytic model, so it is independent of Euclid's postulates. That is, not only did Euclid's proof of I-4 have a flaw, but he could never actually prove that proposition from his axioms. (In fairness, Euclid's work shouldn't be faulted because of insights occurring 2000 years after his work.) ●

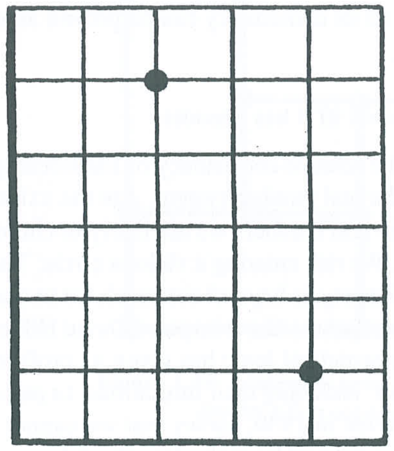

Figure 1.31 The taxicab distance between (2, 5) and (4, 1) is $|2-4|+|5-1|=2+4=6$.

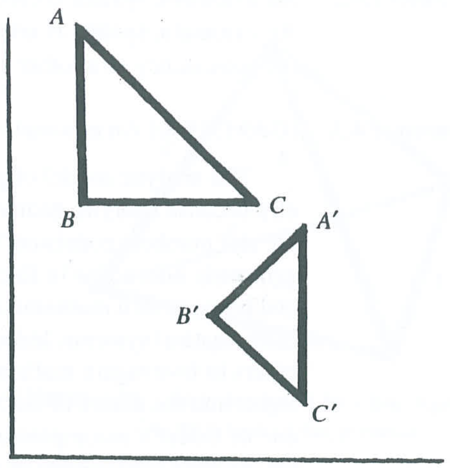

Figure 1.32 In taxicab geometry $\overline{AC}$ is longer than $\overline{A'C'}$, even though $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, and $\angle ABC \cong \angle A'B'C'$.

Example 4 We can stretch a string taut on the surface of a sphere to construct a spherical line segment. Similarly, we can construct a spherical circle by fixing one end of a taut string and swinging the other end around it. Experiment informally on a ball or sphere. Confirm that, within reason, hold in this model. Now draw a spherical line segment with a length of between one-third and one-half the circumference of the sphere ($\overline{AB}$ in Fig. 1.33). Use this line segment to imitate Euclid's construction in Proposition I-1. (See Section 1.3.) Note that the two spherical circles with centers at A and B and radii of length AB do not intersect. This construction does not hold in this model but does in the usual model. Therefore it is independent of Euclid's first four postulates. ●

all five of Euclid's postulates and Playfair's Axiom

For more on taxicab geometry and spherical geometry, see Henderson [14] and Krause [19].

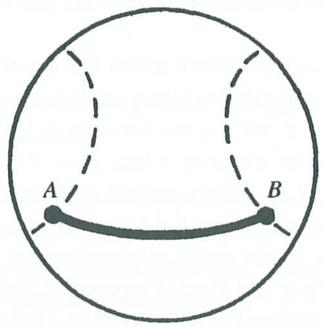

Figure 1.33

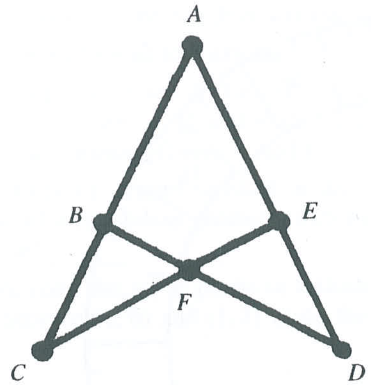

Figure 1.34 Interpret *point* by a dot and *line* by a line segment.

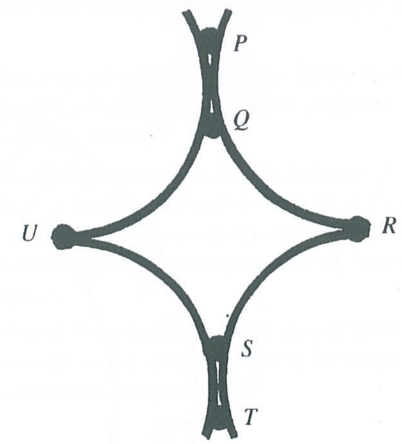

Figure 1.35 Interpret *point* by a dot and *line* by an arc.

Example 5 Hilbert's axiom system is both independent and complete, assuming that it is consistent. (The proof is beyond the level of this text. See the footnote in Section 1.3) ●

Hilbert's axioms are even stronger than being complete: All models of these axioms look exactly alike. In mathematical terminology, all are *isomorphic*. Two more models of the axiomatic system from Example 1 support this idea. The models in Figs. 1.34 and 1.35 both have six points and four lines, as the model in Fig. 1.30 did. They all look different on the surface, but the models in Figs. 1.30 and 1.34 are structurally the same (isomorphic), whereas the one in Fig. 1.35 is different. We can match the points and lines from Fig. 1.30 with those from Fig. 1.34 and have corresponding points on corresponding lines. Indeed, the labeling of these figures provides such a matching. However, no matter how we match the points in Fig. 1.35 with those of these two models, the relation *on* will never be the same. In particular, P and Q share two lines in common, something that never happens in the other models. The model shown in Fig. 1.29 has no chance of being isomorphic with any of these models because it has more points and lines.

Definition 1.4.4 Two models are *isomorphic* iff there is a one-to-one onto function that matches all the elements of one model with those of the other model and that transfers all the relations based on the undefined terms that hold in one model to relations that hold in the other model.

PROBLEMS FOR SECTION 1.4

1. Use graph paper to help you with this problem. (See Example 3.)
   a) Interpret $\overline{AB} \cong \overline{CD}$ in taxicab geometry to mean $d_T(A, B) = d_T(C, D)$. Which of Hilbert's congruence axioms hold in taxicab geometry?
   b) Define and describe "taxicab circles," using the distance d_T. What are the different types of intersection of two different taxicab circles?
   c) Determine whether SSS, ASA, and AAS hold in taxicab geometry.
2. Use a sphere or ball that you can draw on for this problem. (See Example 4.)
   a) Construct triangles of different sizes on a sphere and approximate their angle sum by using a copy of the circle in Fig. 1.36. (Place the center of the circle approximately over the vertex of an angle

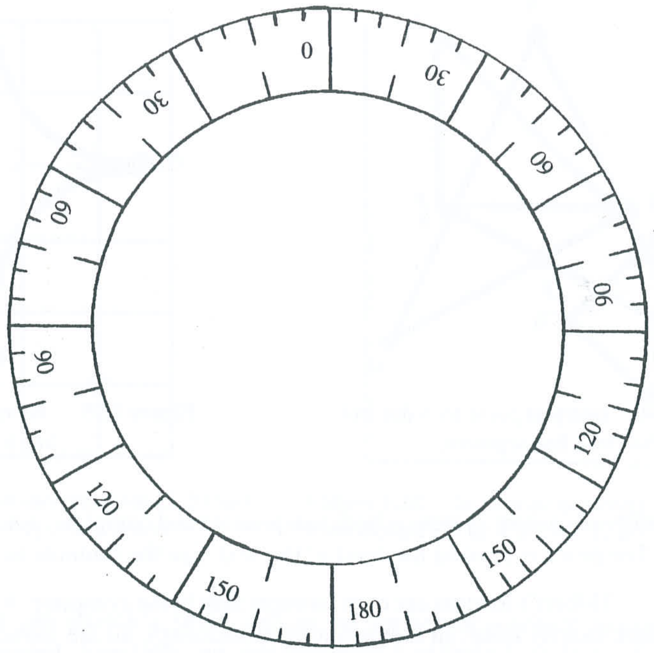

Figure 1.36 A protractor.

and the mark for 0° on one of the sides of the angle.) Does Theorem 1.1.1 hold in this model?

b) Which of Hilbert's congruence axioms hold for the sphere?

c) Determine whether SAS, SSS, ASA, and AAS hold for a sphere.

d) Determine whether the Pythagorean theorem holds for a sphere.

3. Use the axiomatic system of Example 1.

a) What is the smallest positive number of points of a model of this system?

Repeat for lines and explain your answers.

b) Find a model with six points not isomorphic to any of the other models.

c) Find two nonisomorphic models with nine points.

d) Find and prove a theorem of this system.

4. Use the axiomatic system of Problem 6 of Section 1.3.

a) Find several models of this system.

b) Show that axiom (iii) is independent of the other axioms.

c) Repeat part (b) for axiom (iv).

d) Find two nonisomorphic models of this system, each with eight points.

e) Find two nonisomorphic models of the system in Problem 6(c).

5. Use the axiomatic system of Problem 7 of Section 1.3.

a) Find a model with seven points.

b) Find models to show that all the axioms are independent.

6. Consider the axiomatic system with the undefined terms *corner*, *square*, and *on* and the following axioms.

i) There is a square.

ii) Each square is on four distinct corners.

iii) For each square there are four distinct squares with exactly two corners on the given square.

iv) Each corner is on four distinct squares.

a) Show that these axioms are relatively consistent by finding an infinite model in the Euclidean plane.

b) Find a finite model for the first three axioms.

c) Find a finite model for all four axioms.

7. Interpret "Q is between P and R" by $d(P, Q) + d(Q, R) = d(P, R)$, where P, Q, and R are distinct and $d(X, Y)$ is the distance between X and Y.

a) Are all of Hilbert's axioms satisfied in the analytic model of Euclidean geometry with this interpretation?

b) On a graph, color the set of points in taxicab geometry between $(0, 0)$ and $(1, 2)$ using the given interpretation. Which of Hilbert's axioms of order are satisfied in taxicab geometry with this interpretation?

c) In Euclidean geometry, if Q is between P and R and R is between Q and S, then Q is between P and S. Does this property still hold in taxicab geometry with this interpretation of *between*? Explain.

d) If P and R are opposite points on a sphere, which points Q on the sphere would be between P and R under this interpretation?