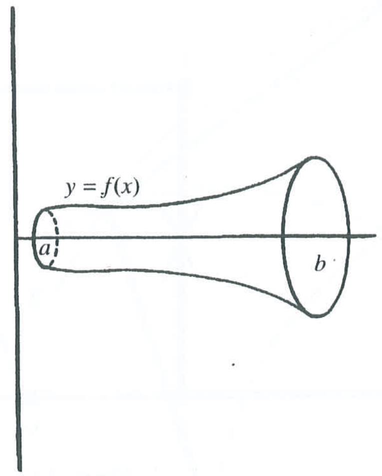


Figure 1.9

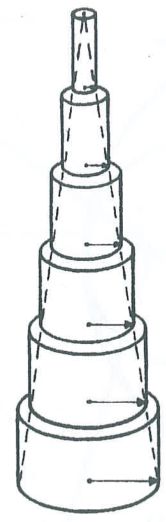


Figure 1.10

1.2 Constructions, Congruence, and Parallels: Euclid's Approach to Geometry

Euclid's masterpiece, *Elements*, exemplifies Greek mathematics; it is an axiomatic system about ideal geometric forms. The modern understanding of axiomatic systems (Sections 1.3 and 1.4) and non-Euclidean geometries (Chapter 3) arose from careful reflection on Euclid's work. Euclid's axiomatic system and these developments have greatly influenced modern mathematics and thus provide ample reason to study Euclid. High school geometry courses, based on Euclid's approach, provide another reason to look at the *Elements* in some detail. In this section we consider constructions, congruence, and parallel lines—all familiar topics of high school geometry found in the *Elements*. In Section 1.5 we consider similarity, another high school topic based on Euclid's work.

Euclid united his own work with that of his predecessors. However, he didn't indicate which of the 465 theorems he discovered, and his text was so successful that no prior geometry text was preserved. Scholars credit Euclid with the organization, the choice of axioms (his postulates and common notions), and some of the theorems and proofs. Euclid sought to achieve Aristotle's goal of starting with self-evident truths and proving all other properties from these assumptions.

1.2.1 Constructions

Euclid's first three postulates and many of his propositions reflect the growth of formal geometry from constructions—figures built from line segments and circles. (See Appendix A for the postulates, definitions and propositions of Book I of Euclid's *Elements*; proposition n of book I is denoted I-n.) Euclid assumed the construction of a line segment given the end points (postulate 1, in Appendix A), the extension of a line segment (postulate 2, in Appendix A), and the construction of a circle given the center

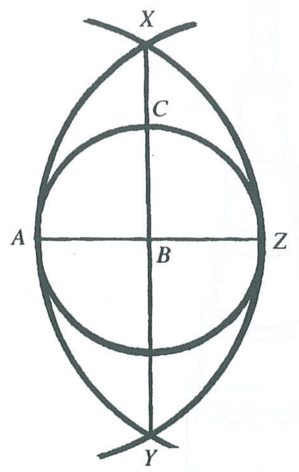


Figure 1.11

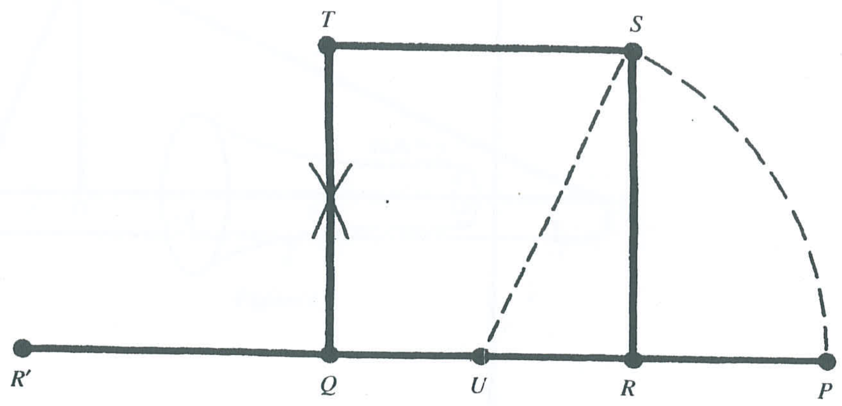


Figure 1.12

and the radius (postulate 3, in Appendix A). By long tradition from Greek to modern times only an unmarked straightedge and a compass can be used in constructions.

Example 1 Construct a square if you know one side of it, say, $\overline{AB}$.

Solution. First construct (following Euclid I-11) $\overline{BC}$, the perpendicular to $\overline{AB}$ at B (Fig. 1.11). Construct the circle with center B and radius $\overline{AB}$ (postulate 3, in Appendix A). Extend $\overline{AB}$ (postulate 2, in Appendix A) until it intersects the circle again, say, at Z. Construct circles with centers A and Z and radii $\overline{AZ}$. Let X and Y be their intersections and construct $\overline{XY}$ (postulate 1, in Appendix A). Explain why $\overline{XY}$ is the perpendicular bisector of $\overline{AZ}$. Let C be an intersection of $\overline{XY}$ and the circle centered at B. Then $\overline{BC}$ is a second side of the square. Construct the rest of the square similarly. ●

Example 2 Construct a regular pentagon if you know one side of it.

Solution. Construct a segment $\overline{PQ}$ such that $PQ = (1+\sqrt{5})AB/2$ (Fig. 1.12). By Problem 5 of Section 1.1, PQ is the length of the diagonal of the pentagon. Let $QRST$ be a square whose sides are the same length as $\overline{AB}$. Find the midpoint U of $\overline{QR}$, using the perpendicular bisector of $\overline{QR}$. Construct the circle with center U and radius $\overline{US}$. One of its intersections with $\overleftrightarrow{QR}$ is the desired point P. The Pythagorean theorem (I-47) shows that $\overline{PQ}$ is as long as claimed.

Fig. 1.13 shows the construction of the pentagon. The circle with center A and radius $\overline{AB}$ intersects the circle with center at B and radius $\overline{PQ}$ at E. Similarly, points C and D are the intersections of circles centered at A and B of appropriate radii. Construct $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{AE}$. Explain why $ABCDE$ is a regular pentagon. ●

Example 3 Construct the tangents to a circle from an outside point.

Solution. Let P be outside the circle with center C (Fig. 1.14). Construct the midpoint M of $\overline{CP}$ and the circle with center M and radius $\overline{MC}$. This circle intersects the original circle in two points, A and B. Then $\overline{PA}$ and $\overline{PB}$ are tangent to the circle through P. Problem 9 provides a way to justify that $\angle PAC$ and $\angle PBC$ are right angles. ●

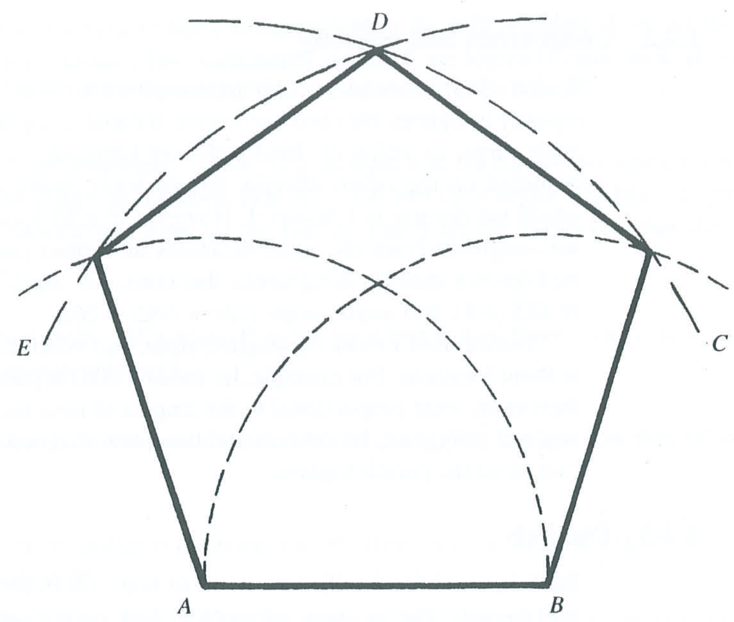


Figure 1.13

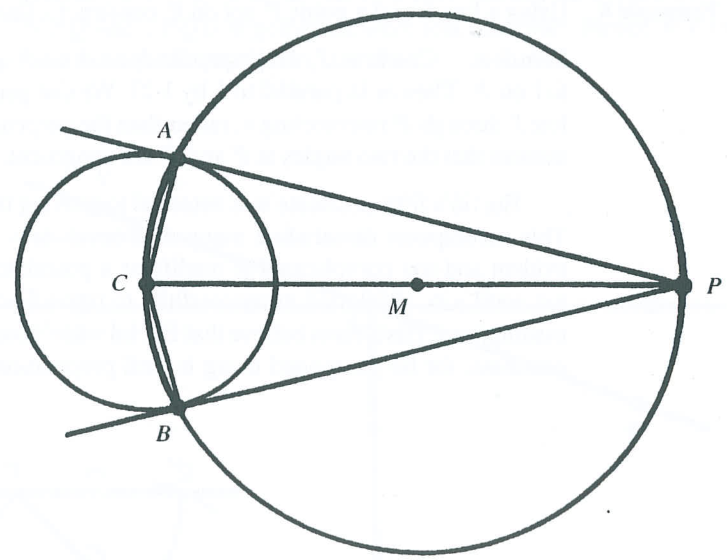


Figure 1.14

1.2.2 Congruence and equality

Euclid's fourth postulate, "All right angles are equal," introduces a second theme, the equality of figures. He used two senses of equal: congruent and equal in measure (length, angle, area, or volume). Intuitively, we know that *congruent* figures coincide if one is placed on the other. Moving figures leads naturally to transformational geometry, which we discuss in Chapter 4. However, Euclid focused on showing that two figures are congruent from the equal measures of various parts. Thus Euclid proves the three well-known triangle congruence theorems *side-angle-side* or *SAS* (I-4), *side-side-side* or *SSS* (I-8), and *angle-angle-side* or *AAS* (I-26).

Euclid didn't think of lengths, areas, and volumes as numbers, so he studied them without formulas. For example, he showed that for parallelograms with the same height, their areas were proportional to the lengths of their bases. Then to compare the areas of any two polygons, he constructed two parallelograms with the same areas (I-45) and compared the parallelograms.

1.2.3 Parallels

Euclid proved the familiar theorems of high school geometry about parallel lines cut by *transversals*, that is, lines intersecting both parallel lines. Euclid's first four postulates allow construction of parallel lines.

Definition 1.2.1 Two lines k and m in the Euclidean plane are *parallel* ($k \| m$) if and only if (iff) they have no points in common or they are equal.

Example 4 Using a line k and a point P not on k, construct a line m parallel to k with P on m.

Solution. Construct l, the perpendicular to k on P, and construct m, the perpendicular to l on P. Then m is parallel to k by I-27. We can generalize this construction with any line l' through P intersecting k, rather than the perpendicular (Fig. 1.15). Proposition I-8 ensures that the two angles at P and Q are congruent, and I-28 guarantees that $m \| k$. ●

Euclid's fifth postulate was essential to proving the most important of his theorems. This assumption dissatisfied many mathematicians because it seemed far from self-evident and too complicated to qualify as a postulate. Many mathematicians from 200 B.C. until A.D. 1800 tried unsuccessfully to prove Euclid's fifth postulate from his other assumptions. Historians believe that Euclid wasn't completely comfortable with his fifth postulate, for he postponed using it until proposition I-29. Playfair's axiom, an easier

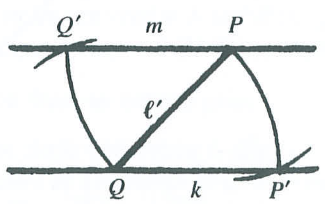


Figure 1.15

statement about parallel lines, is equivalent to the fifth postulate if we accept Euclid's first 28 propositions. (Two statements A and B are logically *equivalent* provided that we can prove both "If A, then B" and its *converse* "If B, then A".)

Euclid's Fifth Postulate That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on the side on which are the angles less than the two right angles.

Playfair's Axiom If a point P is not on a line k, then there is on P at most one line m that does not intersect k.

Exercise 1 Illustrate Euclid's fifth postulate and Playfair's axiom. Explain how they relate to each other.

Theorem 1.2.1 Euclid's fifth postulate is equivalent to Playfair's axiom, assuming that Euclid's first 28 propositions hold.

Proof. (Euclid $\Rightarrow$ Playfair) Suppose that the fifth postulate holds and that we are given a point P not on a line k. Example 4 gives us one parallel, say, $\overleftrightarrow{PR}$, where $\overline{PQ}$ is perpendicular to $\overleftrightarrow{PR}$ and k, as shown in Fig. 1.16. For Playfair's axiom we need to show that $\overleftrightarrow{PR}$ is the only parallel to k through P. We let $\overleftrightarrow{WV}$ be any other line on P with P between V and W. As $\overleftrightarrow{WV}$ is not $\overleftrightarrow{PR}$, either $\angle VPQ$ is acute or $\angle WPQ$ is acute. Then WLOG we assume $\angle WPQ$ to be acute. We now have fulfilled the hypothesis of the fifth postulate: $\angle WPQ$ and $\angle PQU$ together measure less than 180°. Hence $\overleftrightarrow{WV}$ must meet k, showing there is only one parallel to k on P.

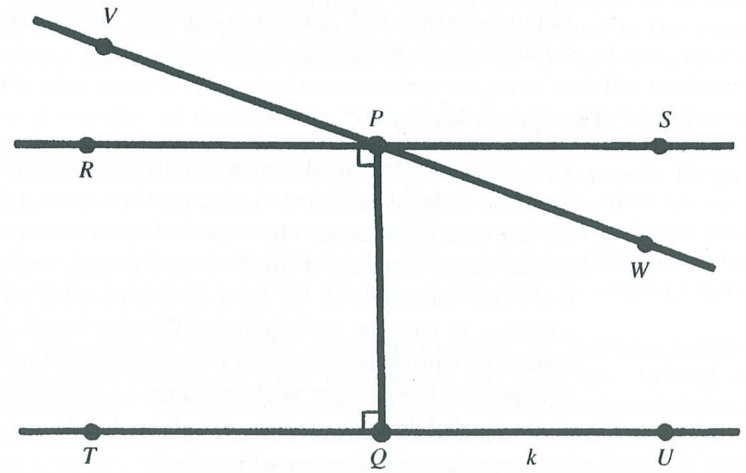


Figure 1.16

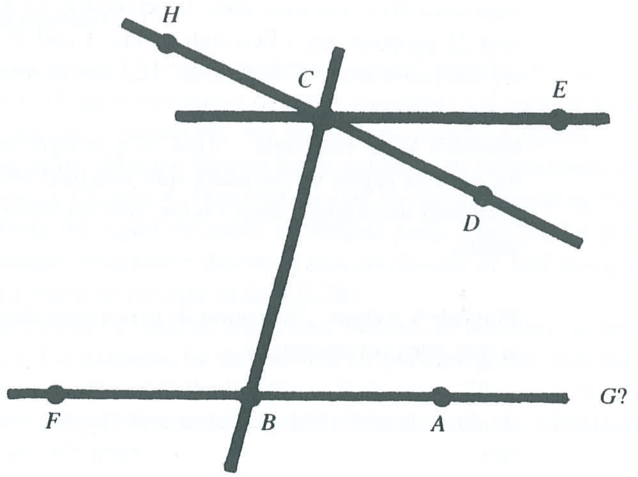


Figure 1.17

(Playfair $\Rightarrow$ Euclid) Suppose that Playfair's axiom holds and $\overline{BC}$ falls on $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ so that $m\angle ABC + m\angle BCD < 180°$. We must show that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ meet on the side of A and D. We construct the line $\overleftrightarrow{CE}$, as shown in Fig. 1.17, such that $\angle BCE \cong \angle CBF$. By Example 4, $\overleftrightarrow{CE} \| \overleftrightarrow{AB}$. Playfair's axiom states that there is only one parallel, meaning that $\overleftrightarrow{CD}$ must intersect $\overleftrightarrow{AB}$, say, at G. However, Playfair's axiom does not tell us directly on which side of B this point G lies. Note that we have a triangle $\triangle BCG$. Euclid I-17 guarantees that the measures of any two angles of $\triangle BCG$ add to less than 180°. Our original assumption about angles $\angle ABC$ and $\angle BCD$ fits I-17 perfectly, but that isn't enough. We must show that the angles on the other side, $\angle FBC$ and $\angle HCB$, do not satisfy I-17. The measures of all four of these angles must add to 360°, and we assumed that the first two add to less than 180°. Hence the last two must add to more than 180°, implying that they are not part of the triangle $\triangle BCG$. Hence G is indeed on the same side of B as A is. ■

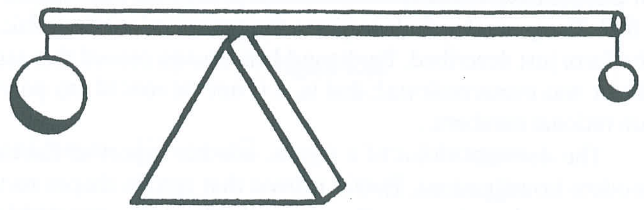
1.2.4 The Greek legacy

Three unsuccessful Greek constructions also inspired the development of mathematics. The first, called *doubling the cube*, was to construct (the side of) a cube with twice the volume of a given cube. The second was *trisecting an angle*. The third, *squaring the circle*, was to construct a square with the same area as a given circle. The Greeks did find exact constructions for these problems using methods beyond a straightedge and compass. In fact, the development of conics (ellipses, parabolas, and hyperbolas) was connected with doubling the cube and trisecting an angle. Related to the third problem, Archimedes, the greatest mathematician of the ancient world, proved that the area of a circle equaled the area of a triangle whose height was the radius of the circle and whose base was the circumference of the circle.

Not until the nineteenth century were mathematicians able to prove that these three constructions were impossible with only an unmarked straightedge and compass. Many

ARCHIMEDES

Archimedes (287–212 B.C.) was the greatest mathematician of the ancient world, and its outstanding engineer and physicist. Many legends attest to his engineering feats, as well as to his absent-mindedness. Upon understanding his law of the lever, as illustrated by the figure, Archimedes is supposed to have claimed, "Give me a place to stand and I will move the world."



Archimedes' law of the lever: The mass of one weight times its distance from the fulcrum equals the mass of the other weight times its distance from the fulcrum.

The story continues with the King of Syracuse asking for a practical demonstration of mechanical advantage. So, by himself, Archimedes pulled a fully loaded ship up a beach by using a sophisticated arrangement of pulleys. Another time the King asked Archimedes to determine, without harming it, if his new crown was made entirely of gold. In a bath Archimedes grasped the principle of the buoyancy of water and so found a solution to the King's problem. In the excitement of discovery, Archimedes ran naked from the bath shouting "Eureka!", meaning "I have found it!" He apparently forgot the world about him when he worked on a problem—drawing figures in ashes or on the oil rubbed on him after a bath. During the extended Roman siege of Syracuse, Archimedes designed various machines that greatly helped in its defense and intimidated the Roman soldiers. He was killed by a soldier when the Romans finally won.

Archimedes brought great imagination and supreme mathematical expertise to his mathematical investigations. He found and proved many theorems, including the exact area of a circle and a parabolic section. His most famous results give the surface area and the volume of a sphere. He also proved theorems concerning tangents and the centers of gravity for various shapes. A number of problems in this chapter concern other results of Archimedes. He brilliantly used the particular geometric properties of each shape he considered. Archimedes' mathematics went beyond his elegant geometric proofs. He gave upper and lower estimates for π. In the "Sand Reckoner" he devised a number system that could handle huge numbers, including his estimate of the number of grains of sand in the earth. In "The Method," a treatise rediscovered in 1906 after being lost for more than a thousand years, he explained how he used his law of the lever to discover new mathematical results that he later proved rigorously.

Archimedes' geometry epitomized Greek mathematical thought. His flawless proofs of difficult theorems were unsurpassed for more than 1500 years. His work also showed the limitations of Greek mathematics. Each result required its own ingenious approach for its proof, unlike calculus and other modern mathematics. Archimedes' writings in physics and mathematics inspired scholars for centuries, especially during the Renaissance. Even today the beauty of Archimedes' mathematics reminds us of what is best in mathematics.

people misinterpret this impossibility, thinking that mathematicians just haven't been clever enough to find constructions. Thus people still propose solutions for these ancient problems. These proofs require abstract algebra and are beyond the scope of this text. (See Gallian [11].) In brief, mathematicians converted the geometric problem of what lengths were constructible with straightedge and compass into an algebraic problem about what irrational numbers could be written in a particular form by using rational numbers, repeated square roots, and arithmetic operations. Both doubling the cube and trisecting an angle involve cubic equations whose roots cannot in general be written in that particular form. Pierre Wantzel proved the impossibility of these constructions in 1837. To square the circle requires the construction of π, which also cannot be written in the form just described. Ferdinand Lindemann proved this assertion in 1882 by proving that π was *transcendental*; that is, it is not the root of any polynomial whose coefficients are rational numbers.

The decomposition of a figure, another aspect of Euclid's geometry, led to other modern investigations. Euclid proved that certain shapes had the same area by decomposing one shape into smaller pieces that could be reassembled to form the other shape. Decomposition puzzles have been popular for centuries, especially the Chinese Tangram puzzle. W. Bolyai in 1832 and P. Gerwien in 1833 independently showed that two polygons in the plane with the same area could be decomposed into one another by using finitely many smaller polygons. (See Boltyanskii [3].) In Chapter 3, we use decomposition to examine area in hyperbolic geometry.

David Hilbert posed the corresponding problem for three-dimensions in a famous talk in 1900, when he presented a list of 23 important, unsolved problems. The same year Max Dehn proved that a regular tetrahedron (triangular pyramid) could not be cut into finitely many polyhedra and then reassembled to form a cube. This and other results showed that a theory of volumes of polyhedra needed limit arguments for rigorous proofs. (See Boltyanskii [3].)

Euclid's *Elements* included results now considered to be part of number theory, algebra, and irrational numbers. Mathematicians learned from Euclid's text for two thousand years, and many important developments in mathematics stem from it. The *Elements* richly deserves its reputation as the most important mathematics book ever written. (See Kline [18, Chapters 4 and 5] for more historical information.)

PROBLEMS FOR SECTION 1.2

1. a) Identify which of Euclid's first 48 propositions (Appendix A) concern constructions.

b) Repeat part (a), replacing *constructions* with *congruence*.

c) Repeat part (a), replacing *constructions* with *equality of measure*.

d) Repeat part (a), replacing *constructions* with *parallels*.

2. a) For lengths a and b, with $a > b$, construct $a + b$ and $a - b$.

b) In Fig. 1.18, $\overleftrightarrow{QS} \| \overleftrightarrow{RT}$. Explain how PQ, PR, PS, and PT are related.

c) Use part (b) to construct the lengths $a \cdot b$ and a/b, given a unit length and the lengths a and b.

3. a) In Fig. 1.19, let $AD = 1$ and $BD = x$. Explain why $CD = \sqrt{x}$.

b) Let M be the midpoint of $\overline{AB}$ and use algebra to show that $CM = AM = BM$ and so A, B, and C are on a circle centered at M.

c) Use segments of length 1 and x to construct a segment of length $\sqrt{x}$.

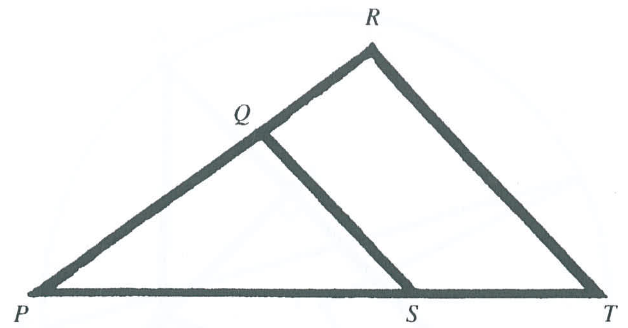


Figure 1.18

d) Construct segment of lengths $\sqrt[4]{x}$ and $\sqrt[8]{x}$. Generalize.

e) Construct a segment of length $\sqrt{1 + \sqrt{2}}$.

4. Begin with a circle and construct the following regular inscribed polygons. You may use earlier constructions in later ones. (An n-sided *polygon*, or *n-gon*, is a set of distinct vertices $P_1, P_2, \ldots, P_n$ in a plane and the *edges* (line segments) $\overline{P_1P_2}$, $\overline{P_2P_3}$, ..., $\overline{P_nP_1}$, with the condition that two edges intersect only at their endpoints. A polygon is *inscribed* in a figure only if the vertices of the polygon are on that figure and the rest of the polygon is in the interior of the figure. A polygon is *regular* only if all the edges and angles are congruent.)

a) An equilateral triangle

b) A square

c) A hexagon

d) An octagon

e) A dodecagon (12-gon)

f) Explain how to inscribe a regular $2n$-gon from one with n sides.

Remarks Carl F. Gauss discovered in 1801 which regular polygons could be constructed with straightedge and compass. He showed that, if the number of sides is a product of a power of 2 and distinct primes of the form $(2^{2^k} + 1)$, the regular polygon is constructible. The only known primes of this form, called Fermat primes, are 3, 5, 17, 257, and 65,537. Gauss actually constructed a regular 17-gon in 1796. He conjectured and Pierre Wantzel proved in 1837 that no other regular polygons are constructible by straightedge and compass alone.

a power of two or

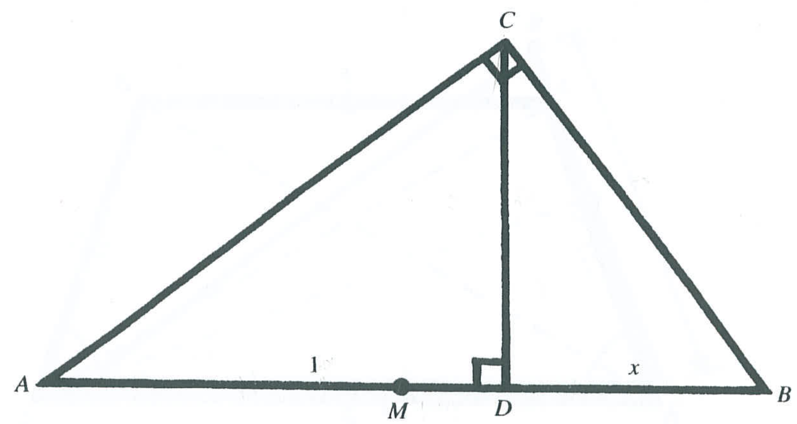


Figure 1.19

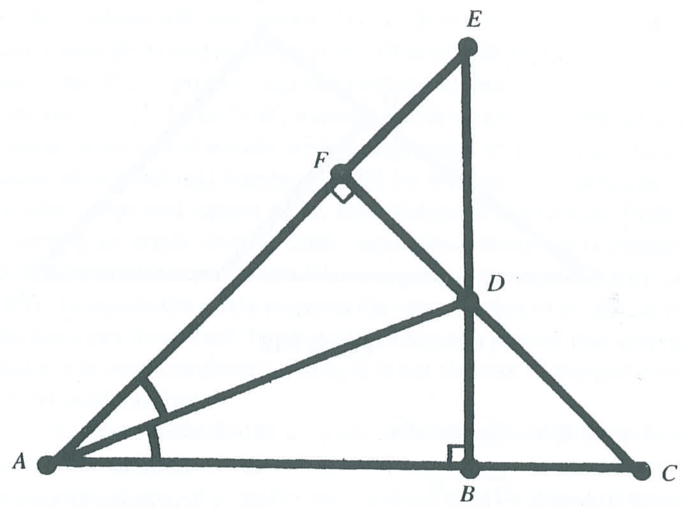


Figure 1.20

5. a) Construct the circumscribed circle for a triangle.

b) Construct the inscribed circle—the circle tangent to the three sides—for a triangle.

6. Use SAS, SSS, and AAS to find and prove the congruence of the four pairs of triangles shown in Fig. 1.20.

7. In Fig. 1.21 assume that $\overline{AD} \cong \overline{BC}$ and $\angle ADC \cong \angle BCD$. Use only Euclid's first 28 propositions in your proofs.

a) Prove that $\angle DAB \cong \angle CBA$. [*Hint:* Find two pairs of congruent triangles formed with the given sides and the two diagonals.]

b) If E is the midpoint of $\overline{CD}$ and F is the midpoint of $\overline{AB}$, show that $\overline{EF}$ is perpendicular to both $\overline{AB}$ and $\overline{CD}$. [*Hint:* Draw $\overline{DF}$ and $\overline{CF}$.]

8. Recall that a *parallelogram* is a quadrilateral whose opposite sides are parallel. Use only Euclid's first 29 propositions in your proofs.

a) Prove that opposite sides of a parallelogram are congruent.

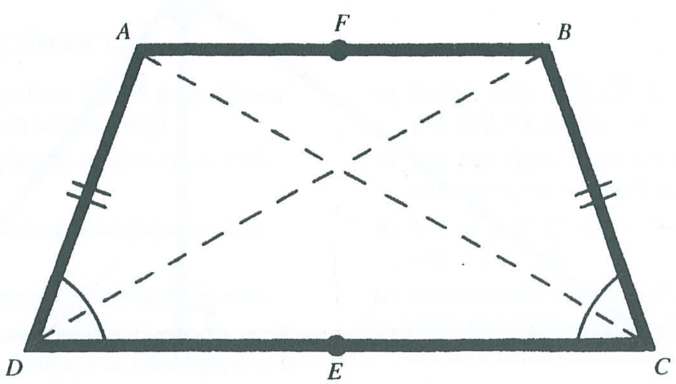


Figure 1.21

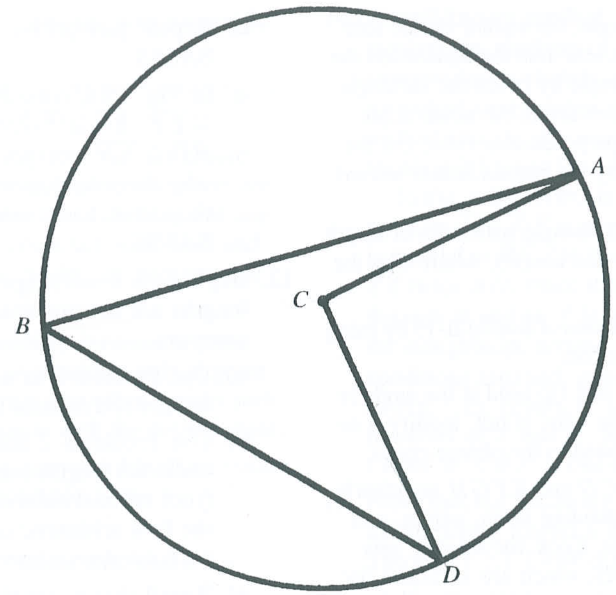


Figure 1.22

b) If $\overline{AB} \| \overline{CD}$ and $\overline{AB} \cong \overline{CD}$, prove that $ABCD$ is a parallelogram. If $\overline{AB} \| \overline{CD}$ and $\overline{AC} \cong \overline{BD}$, must $ABCD$ be a parallelogram? Explain.

c) Prove that a quadrilateral is a parallelogram iff the diagonals bisect each other.

9. Prove that in a circle, the central angle is twice the inscribed angle. That is, in Fig. 1.22, $m\angle ACD$ is twice $m\angle ABD$, where C is the center of the circle. [*Hint:* Draw the diameter through B and use the isosceles triangles $\triangle BCA$ and $\triangle BCD$.]

10. a) Rewrite Euclid II-13 and explain why it is equivalent to the law of cosines for an acute angle. [*Hint:* Examine Fig. 1.23.]

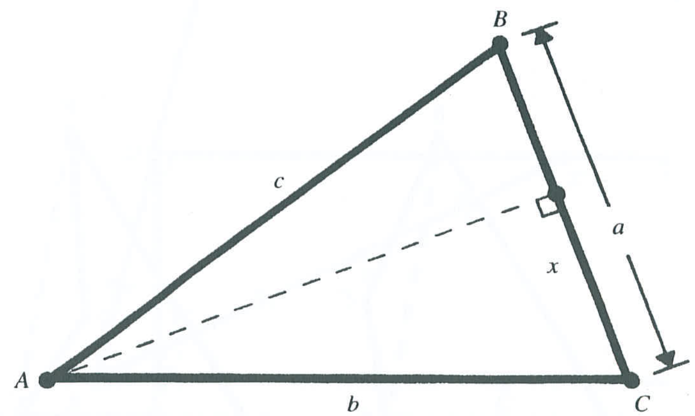


Figure 1.23

Proposition II-13: In a triangle, the square on the side subtending an acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely, that on which the perpendicular falls, and the straight line cut off within by the perpendicular toward the acute angle.

The law of cosines: In any triangle with sides of length a, b, and c, $c^2 = a^2 + b^2 - 2ab\cos(C)$, where C is the angle opposite side c.

b) Prove your reformulation of Euclid II-13 by using the Pythagorean theorem.

c) Does your proof in part (b) hold if the angle is obtuse? If so, explain why; if not, modify it to hold. (Euclid II-12 handles the obtuse case.)

11. Two quadrilaterals $ABCD$ and $EFGH$ are clearly congruent if all corresponding sides, angles, and diagonals are congruent. Look for smaller sets of these correspondences, which are sufficient for convex quadrilaterals.

a) Give an example to show that the congruence of the four pairs of sides (SSSS) is not sufficient. This insufficiency illustrates a basic engineering property: triangles are rigid (SSS); other polygons need triangular bracing to be rigid.

b) A diagonal brace (SSSSD) ensures congruence for convex quadrilaterals. State SSSSD clearly and completely; prove it.

c) State AAAS clearly and completely. Either prove that AAAS is a congruence theorem for convex quadrilaterals or find a counterexample.

d) Repeat part (c) for SASAA, SAASA, and SASSS.

e) In Fig. 1.24, $ABCD$ and $EFGH$ satisfy $\overline{AB} \cong \overline{EF}$, $\overline{BC} \cong \overline{FG}$, $\overline{CD} \cong \overline{GH}$, $\overline{DA} \cong \overline{HE}$, and $\overline{BD} \cong \overline{FH}$, yet they are not congruent. Explain why this situation does not contradict part (b). If they exist, find similar examples for SASAA and SASSS.

12. Begin with a unit length and investigate what other lengths are constructible with a straightedge and compass.

a) Use Problem 2 to describe how to construct integer and rational lengths.

b) Use Problems 2 and 3 to describe how to construct lengths corresponding to numbers built from rational numbers by using square roots and the four arithmetic operations ($+, -, \times, \div$). Call such numbers *constructible*.

c) Recall that a straight line has a first-degree equation $ax + by + c = 0$ and that a circle has a second-degree equation $x^2 + y^2 + dx + ey + f = 0$. Suppose that the coefficients of two lines, two circles, or a line and a circle are all constructible numbers. Explain why the coordinates of the intersections of these lines and circles must also be constructible numbers. [*Hint:* Remember the quadratic formula.]

d) Let (s, t) and (u, v) be the coordinates of two points in the plane. Find the equation of the line through these two points. Explain why the coefficients of this equation can be written in terms of s, t, u, v, and the four arithmetic operations.

e) Repeat part (d) for the circle with center (s, t) and passing through (u, v).

f) Explain why parts (c), (d), and (e) ensure that, if the coordinates of the given points are constructible numbers, the coordinates of any points that you can construct with lines and circles through these points are constructible numbers.

13. Trisecting an angle generally involves a cubic equation. An angle is *constructible* only if given a length, you can construct two other lengths such that the three form a triangle with the desired angle.

a) Show that an angle is constructible iff the cosine of that angle is constructible. Draw a diagram.

b) Use trigonometry to show that $\cos(3z) = 4\cos^3(z) - 3\cos(z)$.

c) If the cosine of a constructible angle is b, show that you can trisect that angle only if you can construct a segment whose length x satisfies the equation $4x^3 - 3x = b$. [Incidentally, $x = \left(\sqrt[3]{b + \sqrt{b^2 - 1}} + \sqrt[3]{b - \sqrt{b^2 - 1}}\right)/2$.]

14. A well-known method for trisecting an angle requires the marking of a particular length on the straightedge, a slightly stronger condition than an unmarked straightedge. Explain why the following method trisects any angle. [*Hint:* Draw $\overline{DM}$, use Problem 3(b), and look for isosceles triangles.]

In Fig. 1.25 $\angle ABC$ is the angle to trisect, D is any point on $\overline{AB}$, $\overline{DE}$ is perpendicular to $\overline{BC}$, and $\overline{DF}$ is parallel to $\overline{BC}$. On the straightedge mark a length YZ twice BD. Place the straightedge so that it goes through B and so Y is on the segment $\overline{DE}$. Slide the straightedge around, keeping the previous two conditions satisfied, until point Z is on the ray $\overrightarrow{DF}$. In Fig. 1.25, points Y' and Z' indicate the correct positions of Y and Z. M is the midpoint of $\overline{Y'Z'}$. Claim: $m\angle CBZ'$ is one-third of $m\angle ABC$.

15. Prove that the following are equivalent to Playfair's axiom, using Euclid's first 28 propositions and Theorem 1.2.1. Draw diagrams.

a) If a straight line cuts one of two parallel lines, it cuts the other.

b) Given two parallel lines and a transversal, the alternate interior angles are congruent (Euclid I-29).

c) If $k \| l$ and $l \| m$, then $k \| m$ (Euclid I-30).

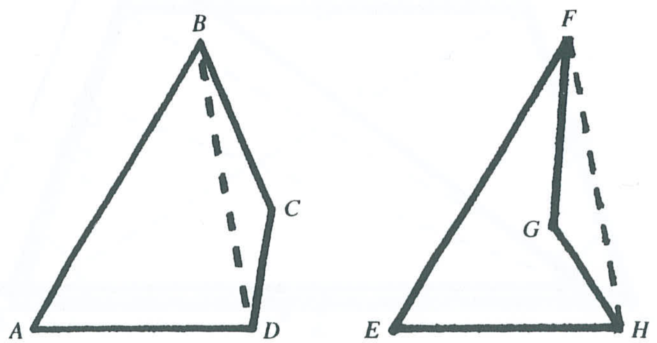


Figure 1.24

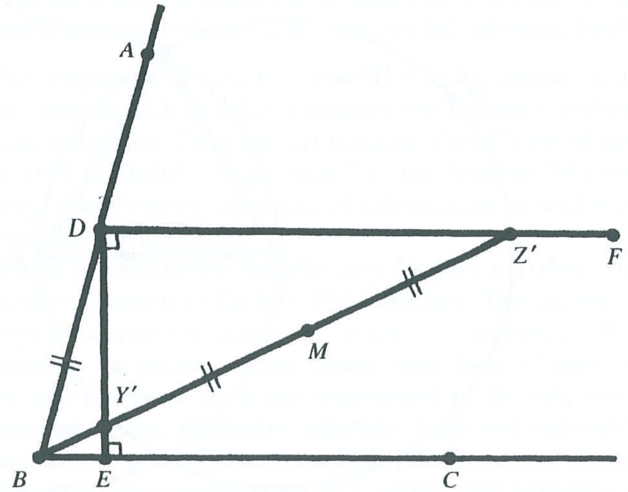


Figure 1.25