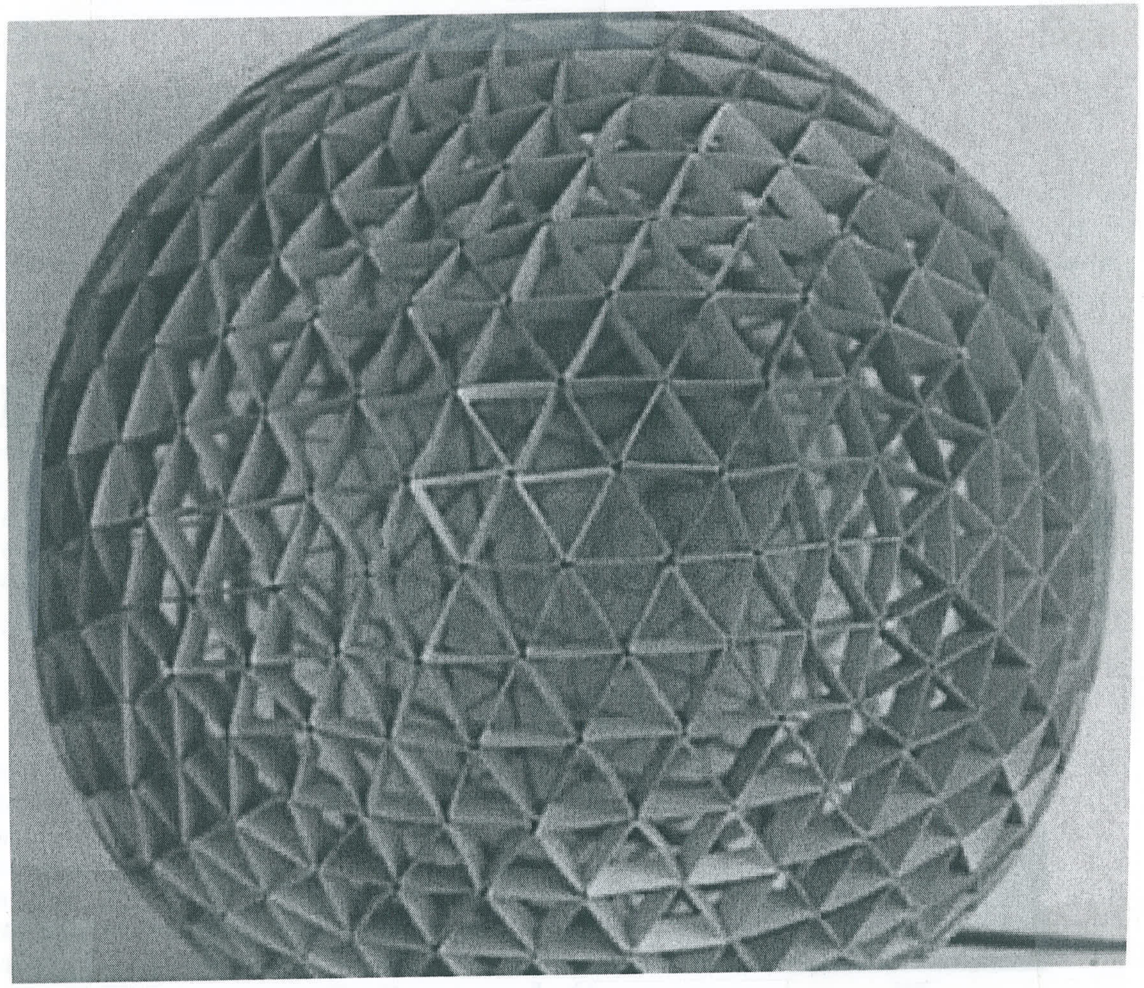
# 1

# Euclidean Geometry



*Geodesic domes, such as the U.S. Pavilion at Expo '67, satisfy pragmatic engineering demands with an elegant geometric shape. Buckminster Fuller used elementary two- and three-dimensional geometry, familiar to mathematicians for more than 2000 years, to fashion the first geodesic domes in the 1950s.*

ΑΓΕΩΜΕΤΡΗΤΟΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ
"Let no one unversed in geometry enter here."
(the inscription Plato is said to have
placed over the entrance to his Academy)

## 1.1 Overview and History

Geometry has a rich heritage as well as contemporary importance. We begin this chapter with a historical examination, focusing on the fundamental contributions of the ancient Greeks. We discuss modern critiques of this foundation and newer geometry stemming from that early work. We also investigate axiomatic systems and models.

Geometric understanding developed in all ancient cultures, consisting largely of geometric patterns and empirical methods for finding areas and volumes of various shapes. The best preserved and most developed pre-Greek mathematics came from Egypt and Babylonia. An Egyptian papyrus dated 1850 B.C. gave an exact procedure for finding the volume of a truncated square pyramid. However, the Egyptians were probably not aware that it was exact or which of their other methods, such as finding the area of a circle, were not exact. A century earlier (by 1950 B.C.) the Babylonians possessed a sophisticated number system and methods to solve problems that we would describe as first- and second-degree equations in one and two variables. The Babylonians, among others, used what we call the Pythagorean theorem. Although Egyptian and Babylonian mathematics dealt with specific numbers rather than general formulas, the variety of examples that survived convinces scholars that these peoples understood the generality of their methods.

### 1.1.1 Euclid, the Pythagoreans, and Zeno

The heritage of deductive mathematics started in ancient Greece and was built on the work of the Babylonians and Egyptians. The Greeks discovered and proved many mathematical properties, including the familiar ones of high school geometry. They also organized this knowledge into an axiomatic system, now known as Euclidean geometry, which honors the Greek mathematician Euclid. We know little about Euclid (circa 300 B.C.) except his mathematics, including the most influential mathematics book of all time: the *Elements* [13]. In it he organized virtually all the elementary mathematics known at the time into a coherent whole. The *Elements* contains definitions, axioms, and 465 theorems and their proofs—but no explanations or applications. For centuries this format represented the ideal for mathematicians and influenced many other areas of knowledge. The Greeks called an axiomatic approach *synthetic* because it synthesizes (proves) new results from statements already known. The Greeks often used a process they called *analysis* to discover new results that they then proved. They analyzed a problem by assuming the desired solution and worked backward to something known. We mimic this procedure in analytic geometry and algebra by assuming that there is an answer, the unknown $x$, and solving for it. In modern times synthetic geometry has



come to mean geometry without coordinates because coordinates are central to analytic geometry.

The Pythagoreans, followers of Pythagoras (580 to 500 B.C.), were among the first groups to focus on theoretical mathematics. Although the Pythagorean theorem had been known at least in numerical form in many cultures, the Pythagoreans are credited with proving this key link between geometry and numbers. The Pythagoreans built their mathematics and their mystical musings on positive whole numbers and their ratios, proportions, and properties. The Pythagoreans developed the theory of positive whole numbers, investigating prime numbers, square numbers, and triangular numbers, among others. They also developed geometric proofs. There is evidence that the Pythagoreans found the proof of Theorem 1.1.1, a theorem as fundamental as the Pythagorean theorem. (See Heath [13, vol. I, 317–320].)

*Notation.* We use the following notation. The *line segment* between two points $A$ and $B$ is denoted $\overline{AB}$, the *length* of $\overline{AB}$ is denoted $AB$, and the *line* through $A$ and $B$ is denoted $\overleftrightarrow{AB}$. The *triangle* with vertices $A$, $B$, and $C$ is denoted $\triangle ABC$, and the *angle* of that triangle with vertex at $B$ is denoted $\angle ABC$. We abbreviate "sum of the measures of the angles" as *angle sum*.

**Theorem 1.1.1** In Euclidean geometry the angle sum of a triangle is 180°.

***Proof.*** Let $\triangle ABC$ be any triangle and construct the line $\overleftrightarrow{DE}$ parallel to $\overline{BC}$ through $A$ (Fig. 1.1). Then (by Euclid's proposition I-29) $\angle DAB \cong \angle CBA$ and $\angle CAE \cong \angle ACB$. Thus the three angles of the triangle are congruent to the three angles $\angle DAB$, $\angle BAC$, and $\angle CAE$ that comprise a straight angle. Hence the angle sum of the triangle equals the measure of a straight angle, 180°. ■

The Pythagorean attempt to ground mathematics on numbers ran into an irreconcilable conflict with their discovery of *incommensurables* (irrational numbers, in modern terms). Commensurable lengths have a common measure—a unit length such that the given lengths are integer multiples of the unit. For example, $\frac{2}{3}$ and $\frac{4}{5}$ have $\frac{1}{15}$ as a common measure. The Pythagoreans proved that the diagonal and side of a square were incommensurable. We say that the diagonal is $\sqrt{2}$ times as long as the side and that $\sqrt{2}$ is not a rational number. Recall that a rational number can be written as a fraction.

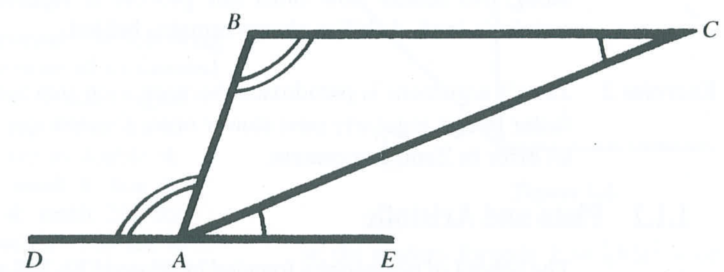



Figure 1.1

**Theorem 1.1.2** No rational number equals $\sqrt{2}$. (The diagonal of a square is incommensurable with the side.)

***Proof.*** Suppose, for a contradiction, that there were two integers $p$ and $q$ such that $p/q = \sqrt{2}$. Without loss of generality (WLOG), assume that $p$ and $q$ are not both even: otherwise we could factor out any common factors of 2. Then $(p/q)^2 = 2$, or $p^2 = 2q^2$. Thus $p^2$ must be an even number, which in turn, forces $p$ to be even. (To illustrate, suppose for a moment that $p$ is odd, say, $p = 2k + 1$. Then $p^2 = 4k^2 + 4k + 1$, an odd number.) If we rewrite the even number $p$ as $2r$, for some integer $r$, then $(2r)^2 = 2q^2$, or more simply $2r^2 = q^2$. As before, we deduce that $q$ must be even. However, $p$ and $q$ are not both even, giving us a contradiction. Thus our initial supposition is invalid, and $\sqrt{2}$ is not a rational number. ■

**Exercise 1** Modify the proof of Theorem 1.1.2 to prove that $\sqrt{3}$ is not rational. Explain where the corresponding argument fails when you try to show that $\sqrt{4}$ is not rational.

Theorem 1.1.2 ruined the Pythagorean's philosophical commitment to explain everything in terms of whole numbers and their ratios. This and other philosophical problems led later Greek mathematicians to base their mathematics on geometry. For example, they no longer thought of lengths, areas, and volumes as numbers because these values could be irrational. The lack of rational numbers for measurement ruled out geometric formulas. Nevertheless, the Greeks made impressive advances in geometry and developed careful, well-founded proofs. The theoretical, abstract nature of Greek mathematics separated it from practical and computational mathematics. However, modern scientists and mathematicians have found important applications of Greek discoveries and the theoretical approach.

Zeno's paradoxes—and the irrationality of $\sqrt{2}$—spurred a careful study of the foundations of geometry. Zeno (circa 450 B.C.) proposed four paradoxes purporting to disprove obvious facts about motion. Zeno's most famous paradox, Achilles and the Tortoise, continues to puzzle people. (See Salmon [26].)

**Achilles and the Tortoise.**
Achilles, the swiftest human, gives a tortoise a head start in a race. Zeno argued that Achilles can never pass the tortoise. For Achilles to catch the tortoise, he must first run to where the tortoise started, but by then the tortoise will have crawled a bit farther. Achilles must now run to this new place, but the tortoise will then be a tiny bit farther along. No matter how often this process is repeated and no matter how small the tortoise's lead, Achilles always remains behind.

**Exercise 2** Zeno's argument is paradoxical because each step seems reasonable, yet we know that faster things regularly pass slower ones. Discuss this paradox, trying especially to find an error in Zeno's argument.

### 1.1.2 Plato and Aristotle

The school of philosophy founded by Plato (429–348 B.C.) next took the lead in the study of geometry. One of Plato's pupils, Eudoxus, developed a theory of proportion and a way to give careful proofs of sophisticated results that applied equally to commensurables



and incommensurables. Furthermore, Eudoxus's work, like limits in calculus, avoided Zeno's paradoxes altogether. Another pupil, Theaetetus, developed a classification of incommensurable lengths and gave the first proof that there are five regular polyhedra.

Plato viewed geometry as vital training for philosophy. He thought that only those who understood the truths of geometry could grasp philosophical truths. In his view, mathematics was certain because it was about ideal, eternal truths, and mathematics was applicable because the physical world was an imperfect reflection of the ideal truth.

Now mathematics is often viewed as part of science with its emphasis on physical reality rather than Plato's ideal view. But mathematics, with its astounding certainty that surpasses any other subject's reliability, seems to have a different content than any science. After all, no one can physically measure $\pi$ to one hundred place accuracy, let alone the more than one billion places that have been found with the aid of computers. Obviously, though, mathematics is not isolated from the real world, as its sophisticated and varied applications reveal.

Aristotle (384–322 B.C.), Plato's most famous student, established his own school of philosophy. Aristotle considered mathematics to be an abstraction of concrete experience. Thus for him the applicability of mathematics derived from its origin in the world. Aristotle thought that mathematics owed its certainty to its careful proofs. He emphasized the necessity of starting with simple, unquestionable truths (axioms or postulates) and carefully proving all other truths from them. His work on logic set the standards of reasoning for two thousand years just as his contributions to many other areas—science, law, ethics and esthetics—profoundly influenced Western culture. (See Kline [18, Chapters 1–3] for more information on ancient, including Greek, mathematics.)

## PROBLEMS FOR SECTION 1.1

In any problems requiring proofs, you may assume any common geometric properties you know, as long as you make your assumptions explicit.

1. The Egyptians used the square of $\frac{8}{9}$ of the diameter for the area of a circle. In measuring a cylindrical granary with a height of 20 ft and a radius of 10 ft, what error in cubic feet and as a percentage would the Egyptians have made? Find the value of $k$ in $(8d/9)^2 = kr^2$. (The Egyptian method effectively approximates $\pi$ by $k$, but it is historically misleading to talk about an "Egyptian value of $\pi$.")
2. Figure 1.2 shows a truncated pyramid. The following is the Egyptian recipe for the volume of a truncated square pyramid with a height of 6 and lengths of 4 at the base and 2 at the top: "You are to square this 4; result 16. You are to double 4; result 8. You are to square 2; result 4. You are to add the 16, the 8, and the 4; result 28. You are to take one third of 6; result 2. You are to take 28 twice; result 56. See, it is 56. You will find it right." Verify that this recipe corresponds to the modern formula $V = \frac{1}{3}h(a^2 + ab + b^2)$. Then derive this formula from $V = \frac{1}{3}HA$ for the volume of a pyramid with height $H$ and base

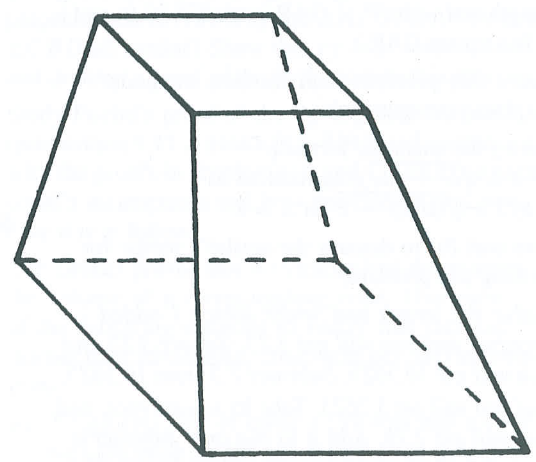


Figure 1.2

area $A$. [*Hint:* A truncated pyramid is the difference between an entire pyramid and a smaller pyramid removed from the top. Use proportions to relate the bases and heights of these two pyramids. Note that $a^3 - b^3 = (a^2 + ab + b^2)(a - b)$.]

A diagram like that shown in Fig. 1.3 appears on a Babylonian tablet, but written in their base 60 notation. Convert these fractions to decimals and discuss what the numbers tell you about Babylonian mathematics.

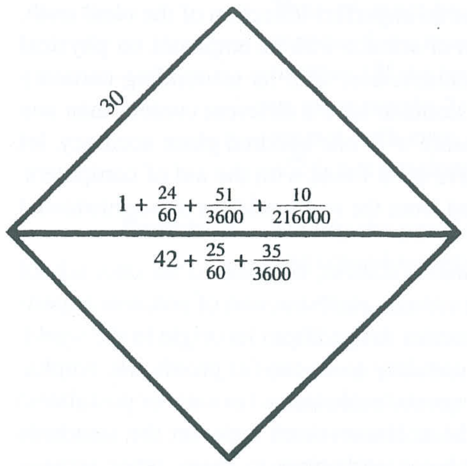



**Figure 1.3**

An (abbreviated) Babylonian problem reads "... I added the length and the width and [the result is] 6.5 GAR. ... [The area is] 7.5 SAR. ... What are the length and width?" (1 GAR is almost 20 ft, and a SAR is a square GAR.)

**a)** Solve this problem with modern methods. Explain your approach.

**b)** Verify the quadratic formula $(-b \pm \sqrt{b^2 - 4ac})/2a$ reduces to $-b/2 \pm \sqrt{(b/2)^2 - c}$ for $a = 1$.

**c)** Use part (b) to discuss the scribe's recipe for solving this problem:

*Halve the length and width which I added together, and you will get 3.25. Square 3.25 and you will get 10.5625. Subtract 7.5 from 10.5625, and you will get 3.2625. Take its square root, and you will get 1.75. Add it to the one, subtract it from the other, and you will get the length and the width. 5 GAR is the length; 1.5 GAR is the width. ... Such is the procedure.*

**5.** The Pythagoreans thought that the pentagram, a pentagon with its diagonals, as shown in Fig 1.4, had mystical qualities.

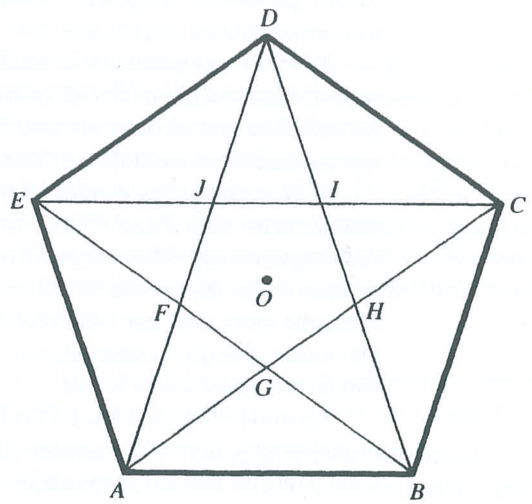



**Figure 1.4**

**a)** Find the measures of the following angles. $\angle AOB$, $\angle OBA$, $\angle ABC$, $\angle BAC$, $\angle AGB$, $\angle CGB$, $\angle ABH$, and $\angle CAD$.

**b)** Verify that $\triangle ABC$ and $\triangle AGB$ are isosceles and similar. List other similar triangles. If $AB = 1$ and $AG = x$, explain why $AC = 1 + x$. [*Hint:* Use $\triangle BCG$.]

**c)** Explain why in part (b) $x$ satisfies $1 + x = 1/x$. Find the exact value of $1 + x$, the diagonal of the pentagon. Verify that the ratio of $AG$ to $GH$ is also $1 + x$. The number $1 + x \approx 1.618$, the *golden ratio,* appears in many natural settings and applications. (See Huntley [16].)

**6.** Assume that any diagonal of a convex polygon is inside it.

**a)** Find the angle sum of a convex quadrilateral, pentagon, and hexagon.

**b)** Find the angle sum of a convex $n$-gon. Prove it with induction.

**c)** What happens if the polygon in part (b) is not convex?



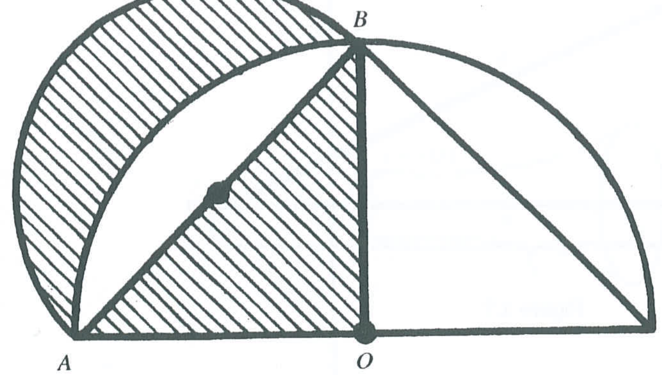
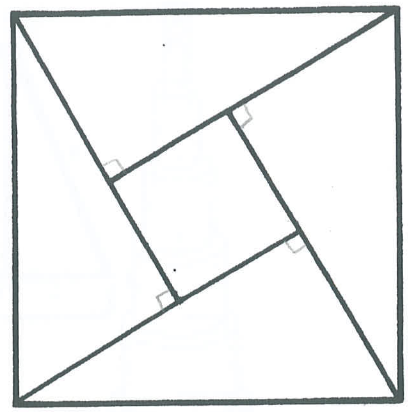



**Figure 1.5**



BEHOLD!

**Figure 1.6**

**7. a)** Find all rectangles with sides of integer lengths whose perimeters equal their areas. (The Pythagoreans considered this problem, but later Greeks did not because they didn't consider areas and lengths as numbers.)

**b)** Find a formula for all rectangles whose perimeters equal their areas. Use this formula to explain your answer in part (a).

**c)** Repeat part (a) for rectangular boxes whose surface areas equal their volumes.

**8.** Eratosthenes (circa 284–192 B.C.) made the most famous and accurate of the Greeks' estimates of the circumference of the earth. He found that at noon on the summer solstice, the sun was directly overhead at Syene, Egypt; at the same time 5000 stadia (approximately 500 mi) north, in Alexandria, Egypt, the sun was $\frac{1}{50}$ of a circle off vertical. Make a diagram, compute the circumference of the earth based on his data and explain your procedure. [*Hint:* Assume that the sun's rays at Syene and Alexandria are parallel lines.]

**9.** In the only surviving Greek mathematical text from before Euclid, Hippocrates (circa 440 B.C.) investigated the areas of *lunes*, which are regions bounded by two circles. Explain his result that the shaded lune shown in Fig. 1.5 has the same area as $\triangle ABO$. [*Hint:* Include the unshaded area between $A$ and $B$.]

**10.** Recall the Pythagorean theorem: In a right triangle, the square on the hypotenuse has the same area as the squares on the sides: that is, $a^2 + b^2 = c^2$. Give a proof of the Pythagorean theorem based on Fig 1.6. The Indian mathematician Bhaskara (1114–1185) devised such a proof but provided only this diagram with the word "BEHOLD!" written below it.

**11.** In Fig 1.7 $\angle ACB$ is a right angle and $\overline{CD}$ is perpendicular to $\overline{AB}$. Why are $\triangle ABC$, $\triangle ACD$, and $\triangle CBD$ all similar? Show that $cy = a^2$ and $cx = b^2$ and develop a proof of the Pythagorean theorem.

**12.** Study Euclid's proof of the Pythagorean theorem (proposition I-47 in Heath [13, 349]) and compare it with the proofs in Problems 10 and 11. Discuss each proof's assumptions and how convincing and how easy it is to follow.

**13.** The Greeks proved that a cylinder has three times the volume of a corresponding cone. Use each of the following methods to verify this fact and discuss their advantages, disadvantages, and assumptions.

**a)** (Empirical) Find or make a cylinder and a cone with the same height and radius (Fig. 1.8). Use the cone three times to fill the cylinder with sand. How close does the volume of sand in the cylinder come to exactly filling it?

**b)** (Calculus) Recall that $\int_a^b \pi (f(x))^2 dx$ gives the volume of revolution generated by rotating

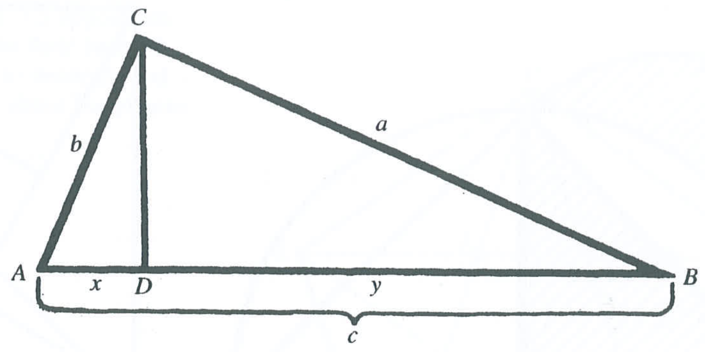



Figure 1.7

around the $x$-axis the curve $y = f(x)$ from $x = a$ to $x = b$ (See Fig. 1.9). Use calculus to compare the volumes of a cone and a cylinder.

**c)** (Stacked disks) You can approximate the volume of a cone with the volume of a stack of disks (Fig. 1.10). If $h$ is the height of the cone and $n$ is the number of disks, the height of each disk is $h/n$. Find a formula for the radius and the volume of the $i$th disk from the top in terms of the radius.

Find the volume of the stack of disks and simplify it to get the approximation formula

$$\text{Volume} \approx \frac{\pi h r^2}{n^3} \sum_{i=1}^{n} i^2.$$

Assume (or better prove by induction) that $\sum_{i=1}^{n} i^2 = (2n^3 + 3n^2 + n)/6$. What happens to this approximation as $n$ approaches infinity?

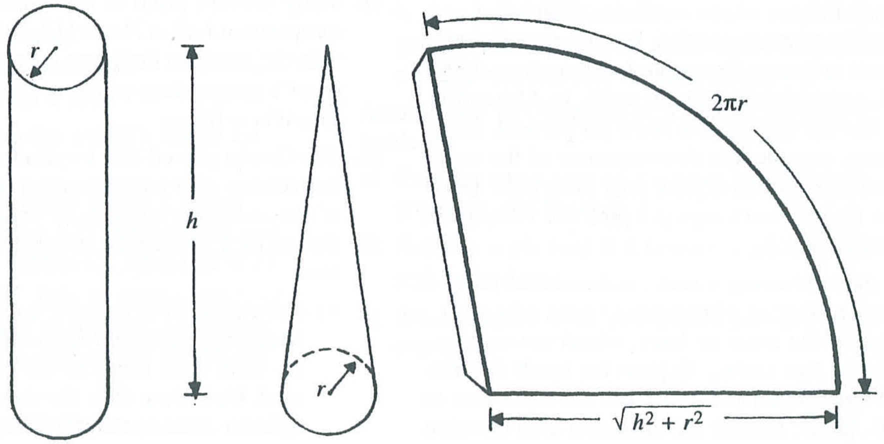



Figure 1.8