EQUIDISTANCE RELATIONS: A NEW BRIDGE BETWEEN GEOMETRIC AND ALGEBRAIC STRUCTURES

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ABSTRACT

This paper investigates the transformations of certain geometric structures into algebraic ones and conversely. The algebraic notion of absolute value corresponds with the geometric one of equidistance. Further, latin squares with absolute values correspond to regular equidistance relations, "near groups" yield 1 point homogeneous equidistance relations, and "near commutative groups" yield 2-point homogeneous equidistance relations.

INTRODUCTION

The synthesis of geometry and algebra has for centuries engendered rich mathematics. This paper presents a new link between these fields by relating absolute values with equidistance relations. We first consider the connection between groups and certain geometries reminiscent of the Cayley graph of a group. This situation generalizes to a much wider class of algebraic and geometric structures, as proven below.

Equidistance relations first appeared in Arthur Cayley's important paper, "A Sixth Memoir on Quantics" [1], in 1859, where he used them to heuristically derive the Euclidean and spherical metrics within the projective plane. Felix Klein's Erlanger Programme [3] of 1872 provides another main link in this paper, namely groups of isometries. Cayley also contributed the notions of the graph of a group and the Cayley numbers, both of which enter into this discussion. See Kline [4, pages 907-923] for a good discussion of both Cayley's and Klein's contributions in these areas. See Sibléy [6] and [7] for further discussion of equidistance relations.

Definitions. Given a metric space (G, d), we can define an equidistance relation E on G by $aa'Ebb'$ iff $d(a, a') = d(b, b')$. However, the four properties in the following definition characterize equidistance relations without reference to a metric. See Sibley [6] for a proof of this. On a set G, E is an *equidistance relation* iff i) E is an equivalance relation on ordered pairs of G, ii) $abEba$, iii) $abEcc$ iff $a = b$, and iv) there are at most $2^{\aleph_0}$ equivalance classes (mod E), where $2^{\aleph_0}$ is the cardinality of the real numbers.

On a metric space (G, d), a bijection f is an isometry iff for all $a, b \in G$, $d(a, b) = d(f(a), f(b))$. Again, the following equilavent definition avoids metrics entirely. A bijection f on G is an *isometry* of E iff for all $a, b \in G$, $abEf(a)f(b)$. Isometries clearly relate quite closely to equidistance relations but they do not in general suffice to determine the metric. See Sibley [6] for conditions under which the isometries determine the metric. Henceforth, a *geometry* (G, E) is a non-empty set G together with an equidistance relation E on G. $I(E)$ is the group of isometries of (G, E). (G, E) is *1-point homogeneous* iff for all $a, b \in G$, there is f in $I(E)$: $f(a) = b$. (G, E) is *2-point homogeneous* iff for all $a, a', b, b' \in G$, there is $f \in I(E)$: $f(a) = b$ and $f(a') = b'$, provided $aa'Ebb'$. Similarly, (G, E) is *n-point homogeneous* iff for all $a_1, \ldots, a_n, b_1, \ldots b_n \in G$; if for all i, j, $a_i a_j E b_i b_j$; then there is $f \in I(E)$: $f(a_i) = b_i$.

We can consider equidistance relations as colored graphs on G by defining two edges aa' and bb' to have the same color iff aa'Ebb'. For each color we thus obtain a graph on G. These various colored graphs form a partition of the complete graph on G. Conversely, any partition of the complete graph on G will define an equidistance relation in the obvious way. (G, E) is *regular* iff each of the colored graphs that E determines on G is regular. That is, every vertex of G has the same number of incident edges of a given color.

Note that for every geometry (G, E), 2-point homogeneity implies 1-point homogeneity which implies regularity.

THE GEOMETRY OF A GROUP

Example. Consider the group $(R, +)$ of the real numbers under addition. We readily define the distance between two points a and b by $/a-b/$. Then aa'Ebb' iff $/a-a'/ = /b-b'/$ defines an equidistance relation.

The situation above generalizes readily for any group $(G, *)$. Define $/a/ = \{a, a^{-1}\}$ and abEcd iff $/a*b^{-1}/ = /c*d^{-1}/$. Recall that the Cayley graph of a group is a directed colored graph on the elements of the group where the edges correspond to multiplication by the elements. Group presentations use only the edges corresponding to the generators of the group. However, if we include all the edges and equate those corresponding to inverses, we obtain the equidistance relation E defined above. See Grossman and Magnus [2] for a discussion of graphs of groups.

Theorem 1. a) If $(G, *)$ is a group, then for the above defined relation E, (G, E) is 1-point homogeneous, provided G has at most $2^{\aleph_0}$ elements. b) Further, if $*$ is commutative, then (G, E) is 2-point homogeneous.

Proof. a) Clearly, E is an equivalence relation since it is defined using =. abEba follows from the group identity $a*b^{-1} = (b*a^{-1})^{-1}$. For the identity, e, $/e/ = \{e\}$. Thus abEcc iff $/a*b^{-1}/ = /c*c^{-1}/ = /e/$ iff $a = b$. The order of G guarantees that there are at most $2^{\aleph_0}$ equivalence classes (mod E). Hence E is an equidistance relation. Define translations t_a: $G \to G$ by $t_a(x) = x*a$ for $a \in G$. These translations are clearly isometries and $t_{(a^{-1}*b)}$ takes a to b. Thus (G, E) is 1-point homogeneous.

b) Suppose $*$ is commutative and that abEcd. Hence either $a*b^{-1} = c*d^{-1}$ or $a*b^{-1} = (c*d^{-1})^{-1} = d*c^{-1}$. In the first case, the isometry $t_{(a^{-1}*c)}$ takes a to c and, after some simple calculations, takes b to d. In the other case we need the central symmetry $s: G \to G$ defined by $s(x) = x^{-1}$. Again, commutativity makes it easy to show that s is an isometry. Then the isometry $t_c \circ s \circ t_{a^{-1}}$ takes a to c and b to d. Hence (G, E) is 2-point homogeneous. Q.E.D.

Note that the set of translations $\{t_a : a \in G\}$ is a subgroup of I(E) which is skew-isomorphic to $(G, *)$: $t_{(a*b)} = t_b \circ t_a$.

Wolff [8] shows that a geometry (G, E) has "parallelograms" iff there is an appropriate operation $*$ on G so that $(G, *)$ is a commutative group. Given $a, b, c \in G$, he defines the fourth point d of the parallelogram by $d = a*c*b^{-1}$ which readily gives both abEcd and acEbd. These are the natural conditions for a parallelogram. While such a point d is definable in any group, either the four points fail to satisfy these conditions or else the choice of d is not unique when $*$ is not commutative.

The connection between the group $(G, *)$ and the translations of (G, E) suggests the following false conjecture: All such 1-point homogeneous spaces are the geometries of groups and all such 2-point homogeneous spaces are the geometries of commutative groups. The example below dispels such a simple characterization. However, the "nearness" of the Cayley numbers to qualifying as a group motivates the direction pursued in the next section.

Example. Let $(C, \circ)$ be the 16 unit elements of the Cayley numbers together with the multiplication of that 8-dimensional algebra over the reals. See Kline [4, page 792] for a discussion of the Cayley numbers. It is well known that this multiplication is neither associative nor commutative. However, the equidistance relation determined from this multiplication using the method above for groups is 2-point homogeneous. A tedious search confirms that none of the 14 groups of order

16 determine an isomorphic geometry. While $(a \circ b) \circ c$ does not always equal $a \circ (b \circ c)$, we do have $/(a \circ b) \circ c/ = /a \circ (b \circ c)/$. Similarly, $/a \circ b/ = /b \circ a/$. These two equations suggest the notions of near groups and near commutative groups defined below.

GENERALIZATION OF THE GEOMETRY OF A GROUP

The definitions of an o-geometry (a geometry with an operation) and of an a-v algebra (an absolute value algebra) characterize those properties needed to generalize the construction above.

Definition. A geometry (G, E) is an *o-geometry* iff i) there is $e \in G$ such that for all $a, b \in G$, there is $c \in G$ with abEec and ii) for all $a \in G$, $\{b: aeEbe\} = /a/$ has at most two elements.

Condition i) guarantees that all distances can be represented as distances from the "origin" and thus allows a natural use of absolute values for distances. Condition ii) provides a means to define inverses and absolute values. For $a \in G$, if $/a/ = \{a\}$, define $a^{-1} = a$. Otherwise, there is a second element in $/a/$ which we define to be a^{-1}. In preparation for the definition of the operation $*$ on G, we first consider a "pre-operation" $*'$. This is a mapping from $G \times G$ into $/G/ = \{/a/ : a \in G\}$. Define $a*'b = /c/$ iff $ab^{-1}Ece$ or equivalently $a*'(b^{-1}) = /d/$ iff abEde. This clearly imitates the definition of absolute value and equidistance in groups. Further, $a*'e = /a/ = e*'a$, $(a^{-1})^{-1} = a$, $a*'a^{-1} = /e/ = a^{-1}*'a$, and $a*'b^{-1} = b*'a^{-1}$ because abEba. This last property corresponds to the group property $(a*b^{-1})^{-1} = b*a^{-1}$. We can convert $*'$ to an operation $*$ on G simply by replacing each occurance of $/a/$ with either a or a^{-1}. However, judicious choices will ensure that e is an identity. Finally, note that inverses are unique since $/e/ = \{e\}$.

Definition. A set G together with a binary operation $*$ is an *a – v algebra* iff i) G has an identity $e: e*a = a = a*e$, ii) each element $a \in G$ has a unique inverse a^{-1} in $G: a*a^{-1} = e = a^{-1}*a$ and for $/x/ = \{x, x^{-1}\}$, iii) for all $a, b \in G$, $/a*b^{-1}/ = /b*a^{-1}/$.

Theorem 2. If (G, E) is an o-geometry, then the pre-operation $*'$ can be converted to an operation $*$ on G so that $(G, *)$ is an a – v algebra. Conversely, given an a – v algebra, by defining abEcd iff $/a*b^{-1}/ = /c*d^{-1}/$, (G, E) is an o-geometry, provided that G has at most $2^{\aleph_0}$ elements.

Proof. The remarks before the definition of an a – v algebra suffice to show that $*'$ can be converted into an appropriate operation $*$ on G. For the converse, note first that E is an equivalence relation because it is defined from equality. Property iii) of an a – v algebra entails abEba. Similarly, the uniqueness of inverses corresponds to the property abEcc iff a = b. The order of G ensures that there are at most $2^{\aleph_0}$ equivalence classes (mod E). Hence E is an equidistance relation. Given $a, b \in G$, let $c = (a*b^{-1})^{-1}$. Then $a*b^{-1} = e*c^{-1}$ and so abEec. This shows property i) of an o-geometry. Finally, the uniqueness of inverses ensures that $\{b: eEbe\} = /a/$ has at most two elements, a and a^{-1}. Q.E.D.

Theorem 3 below extends the above theorem to regular o-geometries and a – v algebras which are latin squares. Recall that a latin square is a set with an operation so that both cancellation laws and both solvability laws hold. Equivalently, every element appears exactly once in each row and column of the multiplication table of that operation.

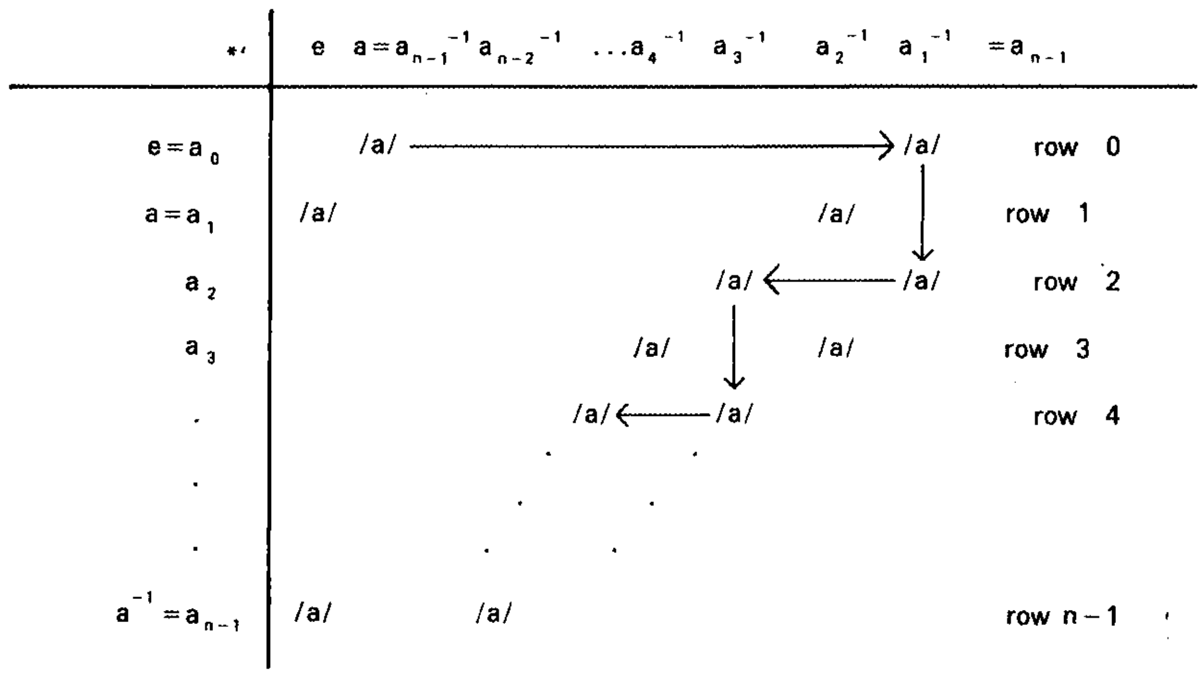
Theorem 3. If (G, E) is a regular o-geometry, then the pre-operation $*'$ can be converted to an operation $*$ on G so that $(G, *)$ is a latin square as well as an a – v algebra. Conversely, if $(G, *)$ is an a – v algebra and a latin square, then (G, E) is a regular o-geometry.

Proof. Given (G, E) a regular o-geometry, $/a/$ appears exactly as many times in each row and column of the table of $*'$ as $/a/$ has elements. The procedure outlined below ensures that $*$ is a latin square. Then we show that e is still an identity under this procedure, the only questionable property to show for an a – v algebra. If $/a/ = \{a\}$, then there is no choice for the replacement of /a/ in the *' table with a in the * table. Further, a will appear exactly once in each row and column. Now suppose that $a \neq a^{-1}$. Thus /a/ appears twice in each row and column. Start at the entry $e*'a =$ /a/ and replace /a/ with a in the * table. Now move horizontally to the next occurrance of /a/ in the *' table at $e*'a^{-1}$. To make (G, *) a latin square, we must take $e*a^{-1} = a^{-1}$, which also makes e a left identity. Now move vertically to the next occurrance of /a/ and replace it with a. Continue alternating horizontal and vertical movements and the placing of a and a^{-1} until you cycle back to the space for e*'a.

unique such cycle by the conditions of regularity and of ii) of an o-geometry. Now eaE $a_i a_{i+1}$ implies that $a_i *' a_{i+1}^{-1} = /a/ = a_i *' a_{i-1}^{-1}$n.

Table (1) summarizes these equalities pictorially which suggests the following argument. Note that the vertical movements change the row by two in this table. Hence if there are an even number of elements in the cycle (n is even), then this process never arrives at the row n–1, an odd number. In this case, we are free to pick the value of $a^{-1}*e$ to be a^{-1} and so complete the table to obtain e as an identity. In the other case, the number n of elements in the cycle is odd. Thus the bottom

TABLE (1) THE PLACING OF /a/ IN THE TABLE OF *'



Since this last movement must be vertical, we again put a in this slot. If in this process we have not encountered the position a*e (and so $a^{-1}*e$), we are free to start again putting $a*e = a$. In this case, we can fill out the whole table for * and have e be a right identity as well. The other possibility is a bit more involved. The cycling process described above corresponds to a cycle of equidistances in (G, E). Let $e = a_0$, $a = a_1, a_2, \ldots, a_{n-1} = a^{-1}$ be the cycle of points in G so that $eaEa_i a_{i+1}$. There is a

row n–1 is even. Hence we will arrive at it in the place $a_{n-1} *' a_{n-1}^{-1}$ which will therefore be assigned the value a in the table for *. From there we come to $a^{-1}*'e$ which will become $a^{-1}*e = a^{-1}$. This in turn implies that a*e must be a. Thus e is indeed an identity. The other properties of an a – v algebra are easily shown to hold.

For the converse, let (G,*) be an a – v algebra and a latin square. We need to show that for all a, b, c ∈ G, the sets {d:abEad} and {f:abEcf} have the same number of elements. That is, for any given distance ab and any two points a and c, the number of points d at that distance from a equals the number of points f at that distance from c. If $/a*b^{-1}/$ has one element, then both of these sets have one element because $(G, *)$ is a latin square. Similarly, if $/a*b^{-1}/$ has two elements, so do these sets. Hence (G, E) is a regular o-geometry. Q.E.D.

Note that different choices of the origin e in (G, E) can result in operations which are not "nearly isomorphic" $(G, *)$ and $(G', *')$ are *nearly isomorphic* iff there is a bijection $f{:}G \to G'$ such that $/f(a*b)/ = /f(a)*'f(b)/$. The example below gives a regular 7 point o-geometry and two quite different operations which have this same geometry from the construction in Theorem 2.

Example. Let (G, E) be the o-geometry given in the pictorial for in Figure (1).

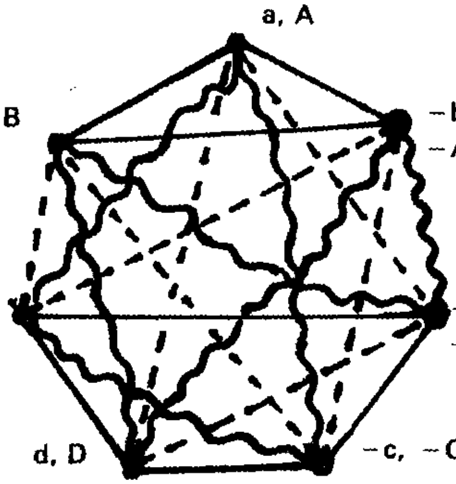


FIGURE (1)

There are three non-zero distances which are represented by solid lines, dashed lines, and wavy lines. To simplify the comprehension of the two multiplication tables, the points are labelled in two ways. The small letters refer to Table (2) in which a is the identity. The capital letters refer to Table (3) in which B is the identity. A minus sign indicates the inverse of an element, except for the identity. The inverses of the underlined elements could have been chosen instead of those values chosen. This however is irrelevant to the fact that the operations $*$ and $*'$ are not nearly isomorphic. To see this, note that $/x*x/ = /x/$ for any x in the first algebra, while $D*'D = -C$ in the second algebra.

TABLE (2) OPERATION * WITH a AS THE IDENTITY

*	a	b	−b	c	−c	d	−d
a	a	b	−b	c	−c	d	−d
b	b	−b	a	$\underline{d}$	$-\underline{d}$	$\underline{c}$	$-\underline{c}$
−b	−b	a	b	$-\underline{d}$	$\underline{d}$	$-\underline{c}$	$\underline{c}$
c	c	$\underline{d}$	$-\underline{d}$	−c	a	$\underline{b}$	$-\underline{b}$
−c	−c	$-\underline{d}$	$\underline{d}$	a	c	$-\underline{b}$	$\underline{b}$
d	d	$\underline{c}$	$-\underline{c}$	$\underline{b}$	$-\underline{b}$	−d	a
−d	−d	$-\underline{c}$	$\underline{c}$	$-\underline{b}$	$\underline{b}$	a	d

TABLE (3) OPERATION *' WITH B AS THE IDENTITY

*'	B	A	−A	C	−C	D	−D
B	B	A	−A	C	−C	D	−D
A	A	−A	B	$-\underline{D}$	$\underline{D}$	$\underline{C}$	$-\underline{C}$
−A	−A	B	A	−C	C	−D	D
C	C	−C	$-\underline{D}$	$\underline{D}$	B	$\underline{A}$	$-\underline{A}$
−C	−C	C	$\underline{D}$	B	$-\underline{D}$	$-\underline{A}$	$\underline{A}$
D	D	−D	$\underline{C}$	$\underline{A}$	$-\underline{A}$	$-\underline{C}$	B
−D	−D	D	$-\underline{C}$	$-\underline{A}$	$\underline{A}$	B	$-\underline{C}$

Next we turn to 1-point and 2-point homogeneous o-geometries.

Definition. (G, *) is a *near group* iff i) (G, *) is an a-v algebra and a latin square, ii) for all a, b, $c \in G$, $(a*b)^{-1} = b^{-1}*a^{-1}$ and * is nearly associative: a rearrangement of paranetheses does not change the absolute value of a product.

Definition. A near group (G,*) is a *near commutative group* iff * is nearly commutative: for all a, $b \in G$, $/a*b/ = /b*a/$.

Theorem 4. a) If (G, *) is a near group with at most $2^{\aleph_0}$ elements, then (G, E) is a 1-point homogeneous o-geometry. b) Further, if (G, *) is a near commutative group, then (G, E) is a 2-point homogeneous o-geometry.

Proof. a) By Theorem 3, we need only find an isometry taking any point a to any other point b. As with groups, define the translation t_c by $t_c(x) = x*c$. Both the property of inverses and near associativity are used to show that t_c is an isometry as follows. $/x*y^{-1}/ = /(x*(c*c^{-1}))*y^{-1}/ = /(x*c)*(c^{-1}*y^{-1})/ = /(x*c)*(y*c)^{-1}/ = /t_c(x)*t_c(y)^{-1}/$. Because (G,*) is a latin square, for all a, $b \in G$, there is (a unique) $c \in G$ so that $a*c = b$ and so $t_c(a) = b$.

b) Given abEcd, we need to find an isometry which takes a to c and b to d. As with commutative groups, the central symmetry $s(x) = x^{-1}$ is an isometry: $/x*y^{-1}/ = /y^{-1}*x/ = /x^{-1}*y/ = /x^{-1}*(y^{-1})^{-1}/^{-1} = /s(x)*s(y)^{-1}/$. Consider the two isometries $t_c \circ t_{(a^{-1})}$ and $t_c \circ s \circ t_{(a^{-1})}$. For both the image of a is c. If $/a*b^{-1}/ = /c*d^{-1}/$ has one element, then they both take b to d because that is the only possibility. If $/a*b^{-1}/ = /c*d^{-1}/$ has two elements, then these isometries take b to different images, one of which is d. Hence (G, E) is 2-point homogeneous. Q.E.D.

Remarks. 1) If (G, E) is a 1-point homogeneous o-geometry, then all the a-v algebras derived from E are nearly isomorphic regardless of the choice of the origin. This is because there is an isometry carrying any choice of the origin to any other and this isometry is the near isomorphism.

2) Part a) of Theorem 4 does not have a converse as the example below shows. The method used to derive this counter-example does not work in the 2-point homogeneous case. Thus, the existence or non-existence of a converse of part b) of Theorem 4 remains an open question.

Example. Let (G, E) be the 8 point geometry given in pictorial form in figures (2) and (3) below. There are five non-zero dis-

FIGURE (2)

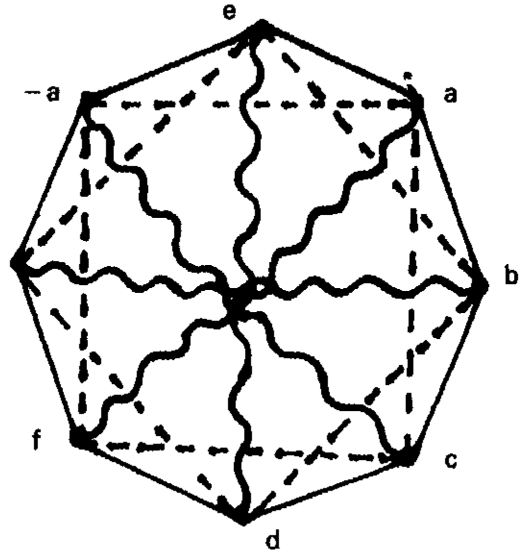
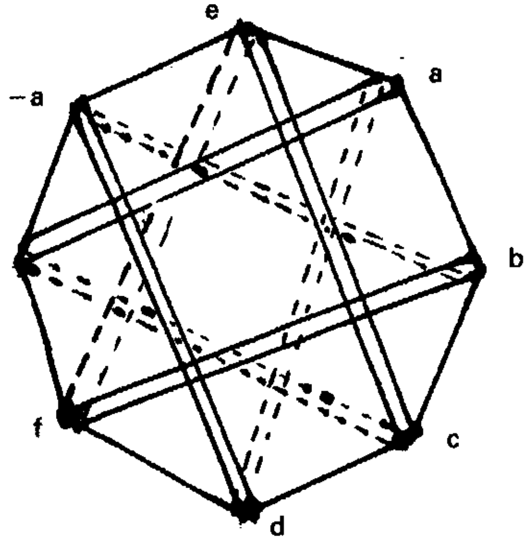


FIGURE (3)



tances represented by solid lines, dashed lines, wavy lines, double solid lines, and double dashed lines. To simplify the comprehension of this geometry, these lines are split between the two figures, except for the solid lines which are in both. Table (4) is an associated multiplication table. The identity is e. The elements c, d, and f are their own inverses, while a and −a are inverses and b and −b are inverses. Note that (a*a)*a = b*a = f but a*(a*a) = a*b = c. Hence * is not nearly associative. Similarly, * is not nearly commutative because a*b = c while b*a = f. No other operation related to E can give a near group either. (G,E) is however 1-point homogeneous since it was constructed by modifying the geometry derived from the dihedral group of the square.

Equidistance relations provided the key motivation in the generalization from groups to near groups to a-v algebras and the related o-geometries. The elementary nature of this material indicates that further and deeper links between geometry and algebra will appear as equidistance relations are better studied.

TABLE (4) MULTIPLICATION TABLE

*	e	a	b	c	d	f	−b	−a
e	e	a	b	c	d	f	−b	−a
a	a	b	c	−b	f	d	−a	e
b	b	f	d	−a	−b	c	e	a
c	c	d	f	e	−a	b	a	−b
d	d	c	−b	a	e	−a	b	f
f	f	−b	−a	b	a	e	c	d
−b	−b	−a	e	f	b	a	d	c
−a	−a	e	a	d	c	−b	f	b

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