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On classifying finite edge colored graphs with two transitive automorphism groups

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Abstract

This paper classifies all finite edge colored graphs with doubly transitive automorphism groups. This result generalizes the classification of doubly transitive balanced incomplete block designs with $\lambda = 1$ and doubly transitive one-factorizations of complete graphs. It also provides a classification of all doubly transitive symmetric association schemes.

The classification of finite simple groups in the 1980s has led to theorems classifying a variety of designs and geometric structures. Edge colored graphs generalize balanced incomplete block designs with $\lambda = 1$ and one-factorizations of complete graphs. This paper classifies the doubly transitive edge colored graphs (abbreviated 2-t ec-graphs), extending results of Kantor [14] and Cameron and Korchmaros [8]. The 2-t symmetric graph designs of Cameron [7] when $\lambda = 1$ match the 2-t ec-graphs for which the number of colors equals the number of vertices. Edge colored graphs are closely related to the rainbows in Aschbacher [2].

Definitions. An *edge colored graph* (V, C) is a finite set V of vertices and a function C from the set E of all undirected edges ab, where $a \neq b$, onto a non-empty set $C(E)$ of *edge colors*. We assume that $|V| \geqslant 2$, where $|V|$ is the number of elements in V. An *automorphism* α of (V, C) is a bijection of V such that for all edges ab and cd, $C(ab) = C(cd)$ iff $C(\alpha(a)\alpha(b)) = C(\alpha(c)\alpha(d))$. An edge colored graph (V, C) is *doubly transitive* iff its group of automorphisms $A(V, C)$ is doubly transitive (2-t) on V.

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Example 1. For any V the 2-t ec-graph obtained by setting $C_M(ab) = 1$ for all edges ab is called the *monochromatic* ec-graph (V, C_M). The 2-t ec-graph obtained by setting $C_T(ab) = ab$ is called the *trivial* ec-graph (V, C_T). Then $A(V, C_M) = A(V, C_T) = S_V$, the symmetric group on V.

Example 2. In a balanced incomplete block designs (BIBD) with $\lambda = 1$, vertex set V, and B the set of blocks, denote by $B(a, b)$ the unique block (line) containing a and b. The edge colored graph (V, C_B) is *derived from* this BIBD if $C_B(ab) = B(a, b)$. Kantor [14] classified all finite 2-t BIBDs with $\lambda = 1$, including affine spaces $AG(n, p^k)$ and projective spaces $PG(n, p^k)$ over the field of order p^k.

Example 3. A one-factorization is an ec-graph where the edges of each color determine a regular graph of degree one. Cameron and Korchmaros [8] classified all finite 2-t one-factorizations and Cameron [6] classified the triply transitive (3-t) ones.

Theorem 1 below classifies the ec-graphs whose automorphism groups are 3-t.

Theorem 1. *If* (V, C) *is a finite doubly transitive edge colored graph and* G *is a group acting triply transitively on* V *with* $G \leqslant A(V, C)$, *then* (V, C) *is monochromatic or trivial with* $|V| \geqslant 2$ *or*

(i) *the doubly transitive one-factorization based on the affine space* $AG(n, 2)$, *where parallel edges are the same color and* $|V| = 2^n$, *or*
(ii) *the doubly transitive one-factorization in* Fig. 1 *and* $|V| = 6$.

Proof. The case $|V| = 2$ is obvious. For $|V| \geqslant 3$ suppose first that there are adjacent edges ab and ax such that $C(ab) = C(ax)$. By 3-t for each $y \in V$ distinct from a and b, there is an automorphism fixing a and b and moving x to y. So for all $y \neq a$, $C(ay) = C(ab)$. In turn, for all $z \neq y$, $C(yz) = C(ya) = C(ab)$, and (V, C) is monochromatic. We may thus assume that adjacent edges are different colors and there are distinct a,

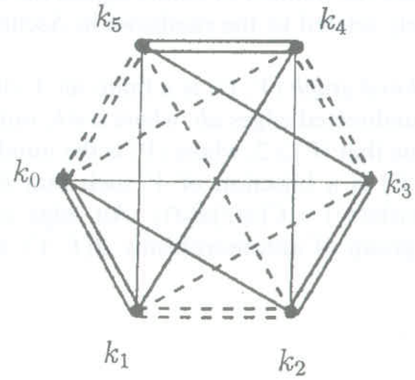

Fig. 1. Different types of lines represent different colors.

b, x and y such that $C(ab) = C(xy)$ since otherwise (V, C) is the trivial ec-graph. For each $z \in V$ distinct from a and b, there is an automorphism fixing a and b and taking x to z, so (V, C) is a one-factorization. Cameron [6] showed that the 3-t one-factorizations are those in (i) and (ii). □

Example 4. If V is a two-dimensional vector space over a field F, $\langle \cdot, \cdot \rangle$ is a non-degenerate, symmetric bilinear form on V, and $C(ab) = \langle a-b, a-b \rangle$, then (V, C) is a 2-t ec-graph. This construction does not generalize to higher dimensional finite spaces because of isotropic elements (see [4]). Any metric space (X, d) becomes an ec-graph by setting $C(ab) = d(a, b)$, and conversely any finite ec-graph becomes a metric space by assigning numbers in [1,2] to the colors.

The first two sections of this paper classify 2-t ec-graphs based on their groups of automorphisms. From the classification of finite simple groups, the finite 2-t groups split into two large collections and one other family. The first collection consists of groups with a 2-t simple subgroup. The 2-t subgroups of some affine group form the second collection. The remaining family consists of groups containing $P\Gamma L(2, 8) = {}^2G_2(3)$ acting on a set with 28 elements (see [14]). Theorem 6, the main result of Section 1, classifies all 2-t ec-graphs whose groups of automorphisms contain some 2-t simple group. In essence, these 2-t ec-graphs are found in Examples 1–3 above and 5–11 below. Theorem 7, which closes Section 1, classifies the 2-t ec-graphs whose automorphism groups contain $P\Gamma L(2, 8)$. Section 2 classifies, to the extent practical, the 2-t ec-graphs whose groups of automorphisms are 2-t subgroups of some affine group. This collection of graphs, which includes those of Example 4, has a far more extensive and complicated structure than those in Section 1, making an explicit counterpart to Theorems 6 and 7 infeasible. Examples 12 and 13 give general constructions for all such 2-t ec-graphs. Section 3 classifies regular 2-t ec-graphs, as in Examples 3 and 4, where the edges of each color form a regular graph on V. It also classifies 2-t point color symmetric graphs and 2-t symmetric association schemes.

1. Non-affine automorphism groups

Here we consider only 2-t ec-graphs whose groups of automorphisms contain a finite simple group or $P\Gamma L(2, 8)$ acting on a set of 28 elements. We present the remaining examples of 2-t ec-graphs with non-affine automorphism groups and lemmas providing the means of determining all of the possibilities. Lemma 2 matches possible 2-t ec-graphs with appropriate subgroups of a 2-t group. Lemma 3 lets us use only the finite simple groups and $P\Gamma L(2, 8)$, rather than all groups containing them.

Example 5. For $V = PG(n, 2)$, each line has three points incident with it. Define $C(E) = V$ and $C(ab) = c$ iff $c \in B(a, b)$, $c \neq a$ and $c \neq b$ (see Fig 2). Then

$A(V,C) = PGL(n+1,2)$ and (V,C) is a 2-t ec-graph as well as a 2-t symmetric graph design with $\lambda = 1$ of Cameron [7]. This construction also applies to the 15-point ec-graph with $A(V,C) = A_7$.

Definition. For edge colored graphs (V,C) and (V,C'), (V,C) is *weaker than* (V,C'), written $(V,C) \preccurlyeq (V,C')$, iff there is a surjection $\gamma : C(E) \to C'(E)$ such that for all ab, $\gamma(C(ab)) = C'(ab)$.

Example 6. Let $V = PG(n,3)$, B its set of lines (blocks), and (V, C_B) the 2-t ec-graph derived from this BIBD with $\lambda = 1$. We define a weaker 2-t ec-graph (V,C) by splitting the six same-colored edges from each line into three pairs as in Fig. 3. More precisely, $C(E) = B \times \{1,2,3\}$ and for each line l in B with any labeling l_i of its points, for $i \in \{0,1,2,3\}$, define $C(l_0 l_i) = (l,i)$ and $C(l_i l_j) = (l,k)$, where i,j and k are distinct elements in $\{1,2,3\}$. Then $A(V,C) = A(V,C_B)$ and (V,C) is a 2-t ec-graph.

Definition. Two edge colored graphs (V,C) and (V',C') are *isomorphic* iff there are bijections $\beta : V \to V'$ and $\gamma : C(E) \to C'(E')$ such that for all a and b, $\gamma(C(ab)) = C'(\beta(a)\beta(b))$.

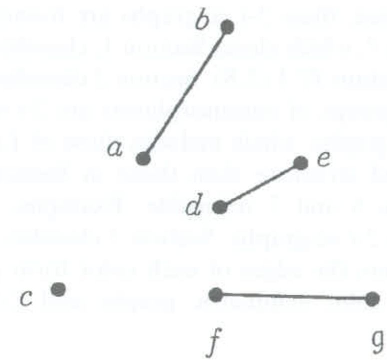

Fig. 2. The vertex c is on the lines on a and b, on d and e, and on f and g. Thus $C(ab) = C(de) = C(fg) = c$.

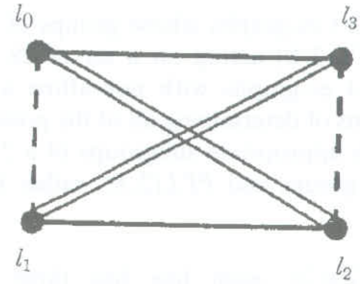

Fig. 3.

Table 1

*	0	1	2	3	4	5
0		1	2	3	4	5
1	1		5	4	2	3
2	2	5		1	3	4
3	3	4	1		5	2
4	4	2	3	5		1
5	5	3	4	2	1	

Example 7. Let $V = PG(n,5)$, B its set of lines, and (V, C_B) the 2-t ec-graph derived from this BIBD with $\lambda = 1$. Analogously to Example 6 we split each color of C_B into five colors with each line colored in (V, C) as in Fig. 1. More precisely, $C(E) = B \times \{1,2,3,4,5\}$. Fix a line k and label its points k_i, where $i \in \{0,1,2,3,4,5\}$. For each $l \in B$, there is $\alpha \in PGL(n+1,5)$ mapping k to l. Because $PGL(n+1,5)$ is 3-t on l the choice of α is immaterial up to isomorphism. Label the points of l as $l_i = \alpha(k_i)$. If we define $C(l_i l_j) = (l, i * j)$, where $i * j$ is given by Table 1, $A(V,C) = A(V, C_B)$, and (V, C) is a 2-t ec-graph.

Example 8. The constructions of Examples 6 and 7 apply to two BIBDs of unitals over the field Z_3 and one BIBD over the field Z_5, yielding three more 2-t ec-graphs, whose automorphism groups are $U_3(3)$, $P\Gamma L(2,8)$, and $U_3(5)$, respectively. Theorem 7 considers $P\Gamma L(2,8)$, which is not a simple group.

Example 9. Let X be the $2n$-dimensional vector space over Z_2 and $G = PSp(2n,2)$, for $n > 2$. We follow the notation in Dixon and Mortimer [11, pp. 245–248]. Now G is 2-t on the subsets Ω^+ and Ω^- of X. For Ω' either Ω^+ or Ω^- and θ_a, $\theta_b \in \Omega'$ with $\theta_a \neq \theta_b$, define $C(\theta_a \theta_b) = a + b$. The transvection switching θ_a and θ_b is in $A(\Omega', C)$. Since the transvections generate G both (Ω^+, C) and (Ω^-, C) are 2-t ec-graphs.

Definitions. For a color c, a *c-chromomorphism* κ is an automorphism such that for every edge ab, if $C(ab) = c$, then $C(\kappa(a)\kappa(b)) = c$; that is, κ preserves the color c, although not necessarily other colors. For a given c the subgroup of all c-chromomorphisms is denoted $K(V,C,c)$, abbreviated $K(c)$. If we are focusing on the color of an edge ab, we write $K(V,C,C(ab))$ or $K(C(ab))$.

Lemma 2 below is a key to using the groups $K(c)$ to generate 2-t ec-graphs. Recall $G_{\{a,b\}}$ is the stabilizer in G of the edge ab. In Lemma 2 the lattice of subgroups K with $A(V,C)_{\{a,b\}} \leqslant K \leqslant A(V,C)$ gives a corresponding lattice of 2-t ec-graphs. The trivial and monochromatic colorings correspond to $A(V,C)_{\{a,b\}}$ and $A(V,C)$, respectively. Note that we may use any edge ab because a 2-t group is transitive on the edges.

Lemma 2. *Let G be a doubly transitive group acting on a finite set V and ab any edge. Then for each subgroup K such that $G_{\{a,b\}} \leqslant K \leqslant G$ there is, up to* orbit is $\{x\}$, which we assume. Then $(V, C_B) \preccurlyeq (V, C)$, where (V, C_B) is derived from the BIBD $PG(3, 2)$. Since $e_{ab} = 2$ in (V, C) any two same colored lines are disjoint. Given this disjointness and the 35 colors of (V, C_B), if $(V, C) \neq (V, C_B)$, then (V, C) would have 7 colors with five lines per color. Suppose that $B(a, b) \neq B(i, j)$ and $C(ab) = C(ij)$. The orbit of ij under G_{ab} determines the lines colored $C(ab)$. The orbit of these lines under G_a determines the seven colors of (V, C). However, direct computation reveals that no automorphism of (V, C) switches a and b, showing (V, C) is not a 2-t ec-graph. Thus (V, C) is either monochromatic or derived from the BIBD.

(iii) Suppose that $G = HS \leqslant A(V, C)$ and $|V| = 176$. As in (i), (V, C) is monochromatic.

(iv) Suppose that $G = Co_3 \leqslant A(V, C)$ and $|V| = 276$. As in (i), (V, C) is monochromatic.

(v) Suppose that $G = Sz(q) \leqslant A(V, C)$, where $q = 2^{2a+1}$ and $a \geqslant 1$, and $|V| = q^2 + 1$. As in (i), (V, C) is monochromatic (see [11,19, p. 250]).

(vi) Suppose that $G = PSL(2, q) \leqslant A(V, C)$, where q is a power of a prime, and $|V| = q + 1$. As in (i), (V, C) is monochromatic.

(vii) Suppose that $G = PSL(n + 1, q) \leqslant A(V, C)$, where q is a power of a prime, $n \geqslant 2$, and $|V| = \sum_{i=0}^{n} q^i$. If x is not incident with $l = B(a, b)$, as in (i), (V, C) is monochromatic. Now suppose that x is incident with l. Because G is 3-t on l, Theorem 1 forces l to be monochromatic. Thus $(V, C_B) \preccurlyeq (V, C)$, where (V, C_B) is derived from $PG(n, q)$. If $(V, C) = (V, C_B)$, we are done. Otherwise there must be some line $B(u, v)$ with $l \cap B(u, v) = \emptyset$ such that $C(uv) = C(ab)$. Then v is not in the plane determined by a, b and u, so as in (i), (V, C) is monochromatic or derived from a BIBD.

(viii) Suppose that $G = PSp(2n, 2) \leqslant A(V, C)$, where $n \geqslant 3$, and $|V| = 2^{2n-1} \pm 2^{n-1}$. As in (i), (V, C) is monochromatic.

(ix) Suppose that $G = U_3(q) \leqslant A(V, C)$, where q is a power of a prime and $q > 2$, and $|V| = q^3 + 1$. As in (i), (V, C) is monochromatic (see [1,12,16]).

(x) Suppose that $G = {}^2G_2(q) \leqslant A(V, C)$, where $q = 3^{2a+1}$ and $q > 3$, and $|V| = q^3 + 1$. As in (i), (V, C) is monochromatic (see [15]). □

Table 2 summarizes the classification in Theorem 6, abbreviating "q is a power of a prime" by "$q = p^m$", "monochromatic" by "mono", and "trivial" by "tr."

Theorem 6. *If (V, C) is a finite 2-t ec-graph and G is a simple group acting doubly transitively on V with $G \leqslant A(V, C)$, then either $G = S_V$ and (V, C) is monochromatic or trivial or else one of the following cases occurs:*

(i) *G is a projective, unitary, or Ree group and (V, C) is the 2-t ec-graph resulting from the unique BIBD on V with $\lambda = 1$ determined by G;*

(ii) *(V, C) is a one-factorization and $G = PSL(2, p)$ for $p = 3$, 5, 7 or 11;*

Table 2

Group	Size of V	All finite 2-t ec-graphs
$PSL(2,11)$	11	mono, tr
A_7	15	mono, tr, BIBD, Ex. 5
HS	176	mono, tr
Co_3	276	mono, tr
$Sz(q)$, $q = 2^{2a+1}, a \geqslant 1$	q^2+1	mono, tr
$PSL(2,q)$, $q = p^m$	$q+1$	mono, tr, Ex. 3, 11
$PSL(n+1,q)$, $q = p^m$	$\sum_{i=0}^{n} q^i$	mono, tr, BIBD, Ex. 5, 6, 7
$PSp(2n,2)$, $n \geqslant 3$	$2^{2n-1} \pm 2^{n-1}$	mono, tr, Ex. 9
$U_3(q)$, $q \neq 2$, $q = p^m$	q^3+1	mono, tr, BIBD, Ex. 8, 10
${}^2G_2(q)$, $q > 3$, $q = 3^{2a+1}$	q^3+1	mono, tr, BIBD, Ex. 8

(iii) *$G = A_7$ and (V,C) is one of the 2-t ec-graphs in Example* 5;

(iv) *$G = PSL(n,p)$ for $p = 2$, 3 or 5 and (V,C) is a 2-t ec-graph in a families in Example* 5, 6, *or* 7;

(v) *$G = U_3(3)$ or $G = U_3(5)$ and (V,C) is one of the 2-t ec-graphs in Example* 8 *or* 10;

(vi) *$G = PSp(2m,2)$ and (V,C) is a 2-t ec-graph in one of the two families in Example* 9;

(vii) *$G = PSL(2,9)$ and (V,C) is the 2-t ec-graph in Example* 11.

Proof. Assume that $e_{ab} = 1$ for any edge ab and there are distinct edges ab and xy with $C(ab) = C(xy)$, since otherwise (V,C) is classified in Theorem 5 or is the trivial graph. From Lemma 4 $e = k_{ab}/2$, e divides $|E|$, and $e \leqslant |V|/2$. Further, $e = |V|/2$ iff (V,C) is a one-factorization. The orbit of the edge xy under the group G_{ab} determines e. If x and y are in the same orbit of size r, the size of the orbit of xy is $r/2$ to ensure $e_{ab} = e_{xy} = 1$. Similarly if x and y are in different orbits, these orbits must be the same size r.

(i) Suppose that $G = PSL(2,11) \leqslant A(V,C)$ and $|V| = 11$. The orbit of x under G_{ab} has three or six elements. Only $r = 6$ is possible, but $e = 1 + r/2 = 4$ does not divide $|E|$. Hence (V,C) is monochromatic or trivial.

(ii) Suppose that $G = A_7 \leqslant A(V,C)$ and $|V| = 15$. As in (i) and Theorem 5(ii) (V,C) is monochromatic, trivial, derived from a BIBD, or Example 5.

(iii) Suppose that $G = HS \leqslant A(V,C)$ and $|V| = 176$. As in (i) (V,C) is monochromatic or trivial (see [8,17]).

(iv) Suppose that $G = Co_3 \leqslant A(V,C)$ and $|V| = 276$. As in (i) (V,C) is monochromatic or trivial (see [8]).

(v) Suppose that $G = Sz(q) \leqslant A(V,C)$ and $|V| = q^2+1$. As in Theorem 5(v) (V,C) is monochromatic or trivial.

(vi) Suppose that $G = PSL(2,q)$, where $q = p^m$, for some prime p, and $|V| = q+1$. Assume that $V = PG(1,q)$. If $p = 2$, G is 3-t and $|V| = 2^m + 1$ is odd, contradicting Theorem 1. If (V, C) is a one-factorization, then $q \in \{3, 5, 7, 11\}$ by Cameron and Korchmaros [8]. Combinatorial restrictions eliminate all other options except $q = 9$, fulfilled in Example 11. Because $PGL(2, 9)$ is 3-t on $PG(1, 9)$, any two such 2-t ec-graphs are isomorphic. Hence (V, C) is monochromatic, trivial, a one-factorization, or Example 11.

(vii) Suppose that $G = PSL(n+1, q)$, where $q = p^m$, for p a prime, $n \geqslant 2$, and $|V| = \sum_{i=0}^{n} q^i$. Assume that $V = PG(n, q)$. Then G_{ab} leaves the line $B(a, b) = B$ stable. The orbit of x under G_{ab} is either V/B or $B/\{a, b\}$. Suppose first that this orbit is V/B. If $B \cap B(x, y)$ is empty, then (V, C) is a one-factorization, contradicting Cameron and Korchmaros [8]. So assume that $\{z\} = B \bigcap B(x, y)$. Then B has just three points because G is 3-t on each line, giving us Example 5. Finally suppose that the orbit of x under G_{ab} is $B/\{a, b\}$. Hence we have a one-factorization of B, and G is 3-t on B. By Cameron and Korchmaros [8] and Cameron [6] $|B|$ is either 4, 6, or 8. The values of 4 and 6 correspond to Examples 6 and 7. If $|B| = 8$, then G would act on B as $PGL(2, 7)$. However, the 3-t ec-graph on eight vertices has $A(V, C) = AGL(3, 2)$. Thus (V, C) is monochromatic, trivial, derived from a BIBD, or one of Examples 5, 6, and 7.

(viii) Suppose that $G = PSp(2n, 2) \leqslant A(V, C)$, $n \geqslant 3$ and $|V| = 2^{2n-1} \pm 2^{n-1}$. Arguments similar to previous ones but somewhat involved force (V, C) to be monochromatic, trivial, or Example 9.

(ix) Suppose that $G = U_3(q) \leqslant A(V, C)$, where q is a power of a prime, $q \neq 2$, and $|V| = q^3 + 1$. As in (i) and Theorem 5(ix), (V, C) is monochromatic, trivial, Example 8, or Example 10 (see [1; 10, p.14; 12; 16]).

(x) Suppose that $G = {}^2G_2(q) \leqslant A(V, C)$, where $q = 3^{2a+1}$, $q > 3$, and $|V| = q^3 + 1$. As in Theorem 5(x), (V, C) is monochromatic, trivial, or derived from a BIBD. □

In addition to the simple groups and the affine family of groups, the 2-t groups include groups containing $P\Gamma L(2, 8)$, acting on a set of 28 unitals, whose 2-t ec-graphs are classified in Theorem 7.

Theorem 7. *If (V, C) is a 2-t ec-graph, where V is the set of unitals for $G = P\Gamma L(2, 8) = {}^2G_2(3) \leqslant A(V, C)$, then*

(i) *(V, C) is monochromatic;*
(ii) *(V, C) is trivial;*
(iii) *(V, C) is derived from the BIBD on V with $\lambda = 1$;*
(iv) *(V, C) is a one-factorization on 28 vertices;*
(v) *(V, C) has, for any color c, $K(V, C, c) \cong PGL(2, 8)$;*
(vi) *(V, C) is the meet of possibilities* (iii) *and* (iv) (*Example* 8); *or*
(vii) *(V, C) is the join of possibilities* (iii) *and* (iv).

Proof. For $a \neq b$ in V we find all K with $G_{\{a,b\}} \leqslant K \leqslant G$. Let $J = PGL(2,8)$, a simple normal subgroup of G with $G_{\{a,b\}} \leqslant J$. The subgroups K such that $G_{\{a,b\}} \leqslant K \leqslant J$ are $G_{\{a,b\}}$, J, $AGL(2,8)$, and T, the eight translations of $AGL(2,8)$ (see [10,6]). These correspond, respectively, to the possibilities (ii), (v), (iv), and (vi) in the theorem. There are at most four more subgroups of $P\Gamma L(2,8)$ whose intersections with J are one of these four subgroups. Three of these potential subgroups actually exist: $P\Gamma L(2,8)$, $A\Gamma L(2,8)$ and the group B for the unique BIBD with $\lambda = 1$ (see [14]). These correspond, respectively, to the possibilities (i), (vii), and (iii) in the theorem. The fourth would have index 2 in B. However, that would entail partitioning the six edges of each line in the BIBD into two sets of three edges, which is not 2-t. □

Remark. The 2-t ec-graph in Theorem 7 (vii) is a 3-factorization of the complete graph of order 28.

2. The affine case

Let V be an n-dimensional vector space over a finite field F with $|F| = p^k$ and $G = A(V,C) \leqslant A\Gamma L(n,p^k)$. Now $A\Gamma L(n,p^k) \leqslant AGL(nk,p)$, so $A(V,C) \leqslant AGL(d,p)$, $d = nk$. In the non-affine case, Lemma 3 let us use only the minimal 2-transitive subgroups because all the related groups contain these minimal groups. Unfortunately, in the affine case the situation is reversed: $A(V,C)$ is a subgroup of the large group, $AGL(d,p)$. Further, as Table 3 illustrates, the often large number of non-isomorphic 2-t ec-graphs for such groups makes a complete classification infeasible. This situation contrasts with the relatively few non-affine examples in Table 3. (The non-simple group $P\Gamma L(2,8)$ accounts for most of the examples when $|V| = 28$.)

We can describe the chromomorphism subgroups in the affine case, although the case $p = 2$ is more complicated than other primes. Because all chromomorphism subgroups $K(c)$ are conjugate, when $p > 2$ we choose $c = C(-vv)$ for a non-zero vector v, and when $p = 2$ we choose $c = C(0v)$. For $p > 2$ Theorem 14 shows that a chromomorphism subgroup $K(C(-vv) = K$ is the semidirect product of its subgroup T_K of "translations" and its subgroup K_0 fixing 0. Thus the construction in Example 12 gives all 2-t ec-graphs for $p > 2$, providing a suitable classification. The "central symmetry" switching each x with $-x$ plays a special role when $p > 2$, so we reserve the letter σ for it. When $p = 2$ Example 13 gives a family of 2-t ec-graphs besides those of Example 12, showing that the analog of Theorem 14 fails. Nevertheless, when $p = 2$ we can construct all 2-t ec-graphs from Examples 12 and 13.

Table 3
The number of non-isomorphic 2-t ec-graphs and non-affine 2-t ec-graphs for selected sizes of V

Size of V	7	8	9	13	16	25	27	28	31	49	64	81
Non-isom.	6	4	8	10	10	⩾18	8	10	12	⩾20	⩾19	⩾28
Not affine	4	2	2	4	2	2	2	10	6	2	2	2

Definitions and notations. Let $G = A(V, C)$ be a 2-t subgroup of $AGL(d,p)$. For $w \in V$ define the *translation* t_w by $t_w(x) = x + w$, and $T = \{t_w : w \in V\}$. Denote the non-zero elements of V by V^*. For $p > 2$ and $v \in V^*$ suppose that $G_{\{-v,v\}} \leqslant K \leqslant G$. Define $T_K = T \cap K$, $V_{T_K} = \{w : t_w \in T_K\}$, $\bar{K}_0 = \{g \in G_0 : \exists t \in T : tg \in K\}$, and $V_{\bar{K}}$ to be the subspace generated by $\{g(v) : g \in \bar{K}_0\}$. For $p > 2$ define $\sigma \in AGL(d,p)$ by $\sigma(x) = -x$ for all $x \in V$. For $p = 2$, replace $G_{\{-v,v\}}$ with $G_{\{0,v\}}$.

Lemma 8. *T is normal in G, and T_K is normal in K; Every $\alpha \in G$ can be written uniquely as $\alpha = tg$, where $t \in T$ and $g \in G_0$; V_{T_K} is a subspace of V; $\bar{K}_0$ is a subgroup of G_0 with $K_0 \leqslant \bar{K}_0$; If $v \in V_{T_K}$, then $V_{\bar{K}} \leqslant V_{T_K}$; If $g \in \bar{K}_0$, then $g(V_{T_K}) = V_{T_K}$ and $g(V_{\bar{K}}) = V_{\bar{K}}$.*

Proof. All of these results are well known or easily shown. □

Example 12. First let $p > 2$, $v \in V^*$, and G be a 2-t subgroup of $AGL(d,p)$. By Lemma 2 if $G_{\{-v,v\}} \leqslant K_0 \leqslant G_0$, then (V, C_{K_0}) is a 2-t ec-graph. Let V_K be the subspace generated by $\{g(v) : g \in K_0\}$. Let V' be any subspace of V such that for all $g \in K_0$, $g(V') = V'$ and $T' = \{t_w : w \in V'\}$. Then $T'K_0 = K$ is a subgroup of G and (V, C_K) is a 2-t ec-graph. For example, $V' = V$ and $V' = \{0\}$ are always stable for any K_0. If $V' = V$, then $K = TK_0$ gives a regular 2-t ec-graph; that is, each color of (V, C_K) is a regular graph. In general, the groups G and K_0 determine which proper subspaces are stable. For any stable V' either $V_K \leqslant V'$ or $V_K \cap V' = \{0\}$. In the first case, the graph is related to the BIBD where V_K is a block (see Figs. 4a and b).

For $p = 2$ every chromomorphism group $K(V, C, C(0v)) = K$ contains the translations t_{u+w} such that $C(uw) = C(0v)$. Thus, $V_K \leqslant V'$. If $T' = \{t_u : u \in V'\}$ and for all $g \in K_0$, $g(V') = V'$, the semidirect product $K = T'K_0$ is a subgroup with $G_{\{0,v\}} \leqslant K$. Then for any 2-t group G and $v \in V^*$ this construction gives a 2-t ec-graph.

For $p > 2$ Theorem 14 below shows that $K = T_K K_0$, showing that Example 12 describes all such 2-t ec-graphs. Lemmas 10–13 consider the types of 2-t subgroups of $AGL(d,p)$, given in Kantor [14].

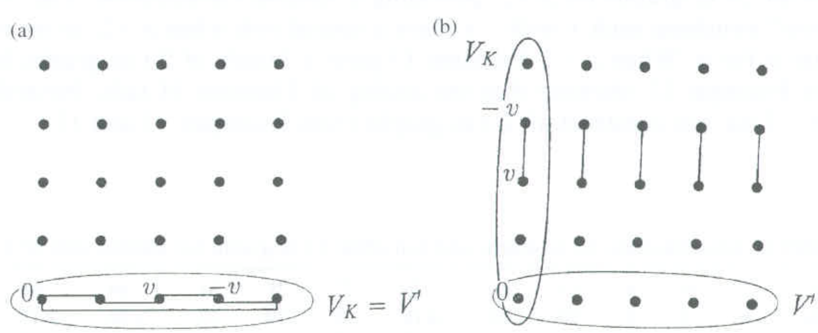

Fig. 4. (a) $v \in V'$, (b) $v \notin V'$.

Lemma 9. *Suppose that* $p>2$, $G=A(V,C)$ *is doubly transitive*, $G\leqslant AGL(d,p)$, $v\in V^*$, $G_{\{-v,v\}}\leqslant K\leqslant G$ *and* $\sigma\in G$. *Then* $K=T_K K_0$.

Proof. Note that $\sigma\in G_{\{-v,v\}}\leqslant K$, σ commutes with all $g\in G_0$ and for $t_u\in T$, $\sigma t_u\sigma = t_{-u}$. For $k\in K$, write $k=t_u g$ with $t_u\in T$ and $g\in G_0$. Then $\sigma(t_u g)\sigma(t_u g)^{-1}=t_{-2u}\in K$. Because p is odd $t_u\in K$. Thus $g=t_{-u}(t_u g)\in K$, and so $t_u\in T_K$ and $g\in K_0$. □

Lemma 10. *Suppose* $G=A(V,C)$ *is a doubly transitive group with* $ASL(n,q^n)\leqslant G\leqslant A\Gamma L(n,q^n)$, $|V|=q^n$, $n\geqslant 2$ *and* $p>2$. *If* $v\in V^*$ *and* $G_{\{-v,v\}}\leqslant K\leqslant G$, *then* $K=T_K K_0$.

Proof. If n is even, then the determinant of σ is 1. Thus $\sigma\in ASL(n,q)\leqslant G$ and $K=T_K K_0$.

Let n be odd and $\kappa\in K=K(C(-vv))$. For $\kappa=tg$ with $t\in T$ and $g\in G_0$, we know $g\in\bar{K}_0$. We need to show that $t\in T_K$, for then $g=t^{-1}(tg)\in K\cap\bar{K}_0=K_0$. Let $u=\kappa(0)=t(0)$ and $w=g(v)$. The action of automorphisms on the subspace $\langle u,v,w\rangle$ forces $t\in T_K$. □

Lemma 11. *If* $G=A(V,C)$ *is a doubly transitive subgroup of* $A\Gamma L(1,p^d)$ *and* $p>2$, *then* $\sigma\in G$.

Proof. If V is the field of order p^d, then $\Gamma_0=A\Gamma L(1,p^d)_0$ is a semidirect product of the cyclic groups V^* and $Aut(V)$. For a a generator of V^* and β the Frobenius map ($\beta(x)=x^p$), the elements $a^i\beta^j$ of Γ_0 satisfy $(a^i\beta^j)(a^k\beta^m)=a^i\beta^j(a^k)\beta^{j+m}$. Because G is 2-t, there is $a\beta^j\in G_0\leqslant\Gamma_0$ mapping 1 to a. Now $(a,\beta^j)^d=\prod_{i=0}^{d-1}\beta^{ij}(a)\in V^*$, which we call a^z. Then $z=\sum_{i=0}^{d-1}p^{ij}=j\left(\frac{p^d-1}{p^j-1}\right)$ has even order in $\mathbf{Z}_{p^d-1}$. Since σ is the only element of order 2 in V^* we have $\sigma\in\langle a^z\rangle\leqslant G$. □

Lemma 12. *Suppose that* $G=A(V,C)$ *is a doubly transitive subgroup of* $AGL(d,p)$, $p>2$, $v\in V^*$, $G_{\{-v,v\}}\leqslant K\leqslant G$, *and* $T_K=\{t_0\}$. *Then* $K=K_0$.

Proof. By the conjugacy of all chromomorphism subgroups, for all $a,b\in V$, $T\cap K(C(ab))=\{t_0\}$. For any edge ab, the p^d translates $(a+w)(b+w)$ of ab all have different colors, so the number of colors is a multiple of p^d. Let $Q_{ab}=\{x:\exists y: C(xy)=C(ab)\}$. By Lemma 4 the number of edges per color divides $(p^d-1)/2$ and so $|Q_{ab}|$ divides p^d-1.

Let $S_{ab}=\sum_{x\in Q_{ab}}x$. Because $|Q_{ab}|$ is relatively prime to p, there is a unique $c_{ab}\in V$ such that $|Q_{ab}|c_{ab}=S_{ab}$. For $\kappa\in K(C(ab))$, $Q_{\kappa(a)\kappa(b)}=Q_{ab}$ and so $K(C(ab))$ fixes c_{ab}. Let $a'=t(a)$ and $b'=t(b)$, where $t(x)=x-c_{ab}$. Then $S_{a'b'}=\sum_{x\in Q_{ab}}t(x)=\sum_{x\in Q_{ab}}(x-c_{ab})=S_{ab}-|Q_{ab}|c_{ab}=0$. Hence $K(C(a'b'))\leqslant G_0$, implying $b'=-a'$. For any $v\in V^*$ by conjugacy $K(C(-vv))=K_0\leqslant G_0$. □

Lemma 13. *Suppose that $G = A(V, C)$ is a doubly transitive subgroup of $AGL(2,p)$, where $p = 5, 7, 11, 19, 23, 29$, or 59 and $SL(2,3) \lhd G_0$ or $SL(2,5) \lhd G_0$. For $v \in V^*$ if $G_{\{-v,v\}} \leqslant K = K(C(-vv)) \leqslant G$, then $K = T_K K_0$.*

Proof. Since $d = 2$ either $T_k = \{t_0\}$, $T_K = T$, or T_K is one-dimensional. If $T_k = \{t_0\}$, use Lemma 12. If $T_K = T$, every subgroup K with $T \leqslant K$ satisfies $K = TK_0$. We split the third case into two subcases: $t_v \in T_K \leqslant K = K(-vv)$ and $t_v \notin T_K$. Because all chromomorphism subgroups are conjugate we may choose $v = (1, -1)$ in the first subcase and $v = (1, 1)$ in the second. By Dixon and Mortimer [11, p. 239], G_0 contains

$$\mu = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

In either case $\mu \in G_{\{-v,v\}} \leqslant K_0$. In the first subcase we have $K = T_K \bar{K}_0$, and so $K = T_K K_0$. In the second subcase we have $\sigma \in G$, so we can use Lemma 9. □

Theorem 14. *If (V, C) is a 2-t ec-graph with $A(V, C) \leqslant AGL(d, p)$ for $p > 2$, $v \in V^*$ and $K = K(C(-vv))$, then $K = T_K K_0$.*

Proof. We use the classification in Kantor [14] of 2-t subgroups of $AGL(d,p)$ and the previous lemmas.

(i) For $G \leqslant A\Gamma L(1, p^d)$ use Lemmas 11 and 9.
(ii) For $ASL(n, p^d) \leqslant G$, where $n \geqslant 2$, use Lemma 10.
(iii) For $Sp(n, p^d) \leqslant G$, use Lemma 9 because $\sigma \in Sp(n, p^d)$.
(iv) For $SL(2,3) \lhd G_0$ or $SL(2,5) \lhd G_0$ and $|V| = p^2$, $p = 5, 7, 11, 19, 23, 29$ or 59, use Lemma 13.
(v) By Aschbacher [3] the three remaining groups with $|V| = 3^d$, $d = 4$ or $d = 6$ contain σ, so use Lemma 9. □

The classification of 2-t affine groups in Kantor [14], Theorem 14, and Example 12 provides a method, even if burdensome, to construct any affine 2-t ec-graph when $p > 2$. Given G a 2-t subgroup of $AGL(d, p)$ and any $v \in V^*$, find all K_0 such that $G_{\{v,-v\}} \leqslant K_0 \leqslant G_0$, all subgroups T' of translations, and all subspaces left stable by $K = T'K_0$. From G, K_0, and T' Example 12 gives any 2-t affine ec-graph.

We turn now to the case $p = 2$ which admits the more complicated construction in Example 13.

Example 13. Let V be the field of order 2^{kn} and F the subfield of order 2^k, where $1 < k, n$. The elements of $G = AGL(1, 2^{kn})$ can be written as $g_{a,b}$, where $g_{a,b}(x) = ax + b$ and $a \in V^*$, $b \in V$. Note that $t_b = g_{1,b}$ and $G_{\{0,1\}} = \{t_0, t_1\}$. Let w generate V^*, $j = (2^{kn} - 1)/(2^k - 1)$ and $u = w^j$. Then u generates F^*. Let $K = K(V, C, C(01))$ be the subgroup generated by $g_{u,w}$ and the translations $\{t_q : q \in F\} = T_K$. Then

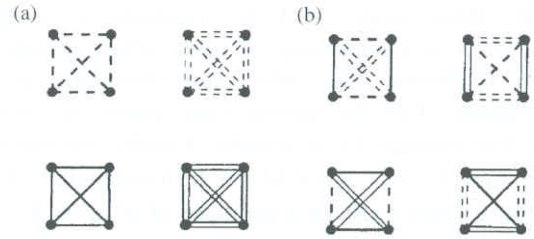

Fig. 5. (a) Four parallel lines in the BIBD with 16 vertices and 4 vertices per line. (b) Coloring of the edges of the four lines from (a) following the construction in Example 13.

$G_{\{0,1\}} \leqslant T_K \leqslant K$. The elements of K are of the form $g_{u^t,z+q}$, where $q \in F$. Thus, K is not the semidirect product $T_K K_0$. If $L = T_K \bar{K}_0 = \{g_{u^t,q} : q \in F\}$, then $|K| = |L|$. Further, (V, C_L) is derived from the BIBD in which F is one line. Both (V, C_K) and (V, C_L) are 2-t ec-graphs with the same translations in $K(C_K(ab))$ and $K(C_L(ab))$, but edges in one line of (V, C_L) are split up among translates of those edges to form (V, C_K). Fig. 5a and b illustrate this situation.

Example 13 complicates finding all 2-t ec-graphs when case $p = 2$. Fortunately, if K satisfies $G_{\{0,v\}} \leqslant K \leqslant G$, where G is a 2-t subgroup of $AGL(d,2)$ and $v \in V^*$, then $L = T_K \bar{K}_0$ satisfies $|K| = |L|$ and $K \bigcap L = T_K K_0$, which we call J. Lemma 8 ensures that $G_{\{0,v\}} \leqslant J \leqslant L$. Clearly, (V, C_K), (V, C_L) and (V, C_J) are 2-t ec-graphs with $(V, C_J) \preccurlyeq (V, C_K)$ and $(V, C_J) \preccurlyeq (V, C_L)$. Further, $T_K = T_L = T_J$. The set of edges of one color for either (V, C_K) or (V, C_L) is a union of $|\bar{K}_0|/|K_0|$ families of same colored edges of (V, C_J). As in Example 13, the families forming one color in (V, C_K) are translates of the families forming one color in (V, C_L). The constructions of Example 12 and this generalization of Example 13 are the only ones possible when $p = 2$.

3. Regular edge colored graphs

As mentioned in Example 4, we can convert a metric space to an ec-graph. Regular ec-graphs correspond to metric spaces in which the configuration of distances from any point to other points is independent of the point. A geometrically interesting and more restricted family of metric spaces are those with transitive isometry groups, as defined below. We classify the 2-t ec-graphs corresponding to both of these situations as well as symmetric association schemes.

Definition. An edge colored graph (V, C) is *regular* iff for each color c, the edges ab such that $C(ab) = c$ form a regular graph on V.

Theorem 15. *If (V, C) is a finite regular 2-t ec-graph, then either (V, C) is monochromatic; (V, C) is a one-factorization; $|V| = 28$ and $A(V, C) = P\Gamma L(2, 8)$; or $|V| = p^d$, $A(V, C) \leqslant AGL(d, p)$, and for any edge ab, $T \leqslant K(V, C, ab)$.*

Proof. We consider three cases: $A(V,C)$ contains a simple 2-t group, it contains $P\Gamma L(2,8)$, or it is a subgroup of some affine group $AGL(d,p)$. In the first case Theorem 6 shows that only monochromatic ec-graphs and one-factorizations are regular. For the second case, parts (i), (iv), (v), and (vii) of Theorem 7 list the four regular 2-t ec-graphs. Finally, for the third case suppose $A(V,C) \leqslant AGL(d,p)$ and $|V| = p^d$. First let $p > 2$. In order that (V,C) be regular, the number of edges of any color must be a multiple of p^d. For $K = K(V,C,C(-vv))$, the number of edges of the color $C(-vv)$ is $|K|/|G_{\{-v,v\}}| = |T_K||K_0|/|G_{\{-v,v\}}|$. Now (V, C_{K_0}) is a 2-t ec-graph with $|K_0|/|G_{\{-v,v\}}|$ edges of color $C_{K_0}(-vv)$. In Lemma 12 we showed that $|K_0|/|G_{\{-v,v\}}|$ is relatively prime to p^d. Hence $|T_K| = p^d$ and $T_K = T$. A similar argument holds when $p = 2$ once we substitute $\bar{K}_0$ for K_0 and G_{0v} for $G_{\{-v,v\}}$. □

Example 14. Let V be the d-dimensional vector space over Z_p, $G = AGL(d,p)$, ab any edge in V and $H = TG_{\{a,b\}}$ (or TG_{ab} if $p = 2$). Now T is the smallest transitive subgroup of G, and so H is the smallest transitive subgroup containing $G_{\{a,b\}}$ (or G_{ab} if $p = 2$). Thus all regular 2-t ec-graphs (V,C) satisfy $(V,C_H) \preccurlyeq (V,C)$.

Remark. T and V are isomorphic as groups, so (V,C_H) corresponds to the equidistance relation on V defined in Sibley [18].

From Example 14 the regular affine 2-t ec-graphs correspond to the subgroups K such that $TG_{\{-v,v\}} \leqslant K \leqslant G$ (or $TG_{0v} \leqslant K \leqslant G$, if $p = 2$). What more can we say about these graphs? The number of colors of (V,C) must divide the number of colors of (V,C_H) in Example 14, which is $\frac{p^d-1}{2}$ if $p > 2$ and $p^d - 1$ if $p = 2$. Example 15 shows that all such divisors are possible. Unfortunately, Example 16 shows that there are non-isomorphic 2-t ec-graphs for some divisors.

Example 15. Let V be the field of order p^d and $G = AGL(1,p^d)$. The cyclic group $G_0 = V^*$ has $p^d - 1$ elements. For each divisor j of $p^d - 1$ there is a unique subgroup J_j of V^* containing j elements. If $p > 2$, assume that j is an even divisor; if $p = 2$, j can be any divisor. If we use $K_j = TJ_j$ in Lemma 2, then (V, C_j) has $(|V| - 1)/j$ colors.

Example 16. Let V be the two-dimensional vector space over Z_5. There are non-isomorphic regular 2-t ec-graphs (V,C) and (V,C') with $C(E) = C'(E) = \{0,1,3\}$. Call the six classes of parallel lines B_m, for $m \in Z_5 \cup \{\infty\}$, where the lines $y = mx + c$, for m, $c \in Z_5$ are in B_m and the lines $x = c$ are in B_∞. Define

$$C(ab) = \left\{ \begin{array}{ll} 0 & \text{if } B(a,b) \in B_0 \cup B_\infty \\ 1 & \text{if } B(a,b) \in B_1 \cup B_4 \\ 3 & \text{if } B(a,b) \in B_2 \cup B_3 \end{array} \right\}$$

and

$$C'(ab) = \begin{Bmatrix} 0 & \text{if } B(a,b) \in B_0 \cup B_\infty \\ 1 & \text{if } B(a,b) \in B_1 \cup B_2 \\ 3 & \text{if } B(a,b) \in B_3 \cup B_4 \end{Bmatrix}.$$

Since $A(V,C)$ has 4800 elements and $A(V,C')$ has only 2400 elements (V,C) and (V,C') are not isomorphic. (For each color c some c-chromomorphisms switch the other two colors, so $K(V,C,c)$ and $K(V,C',c)$ each have index 6 in $A(V,C)$ and $A(V,C')$, respectively.)

Theorem 16 classifies the 2-t ec-graphs with a transitive group of isometries, a stronger condition than regular.

Definitions. An automorphism ρ of (V,C) is an *isometry* iff $C(ab) = C(\rho(a)\rho(b))$ for all edges ab. Denote the group of isometries by $I(V,C)$. An ec-graph (V,C) is *point color symmetric* iff $I(V,C)$ is transitive on V and $A(V,C)$ is transitive on the colors $C(E)$. (See Chen and Teh [9] for more on point color symmetric graphs.)

Theorem 16. *A 2-t ec-graph is point color symmetric iff it is a doubly transitive symmetric association scheme iff it monochromatic or regular and affine or item* (*v*) *of Theorem* 7.

Proof. A 2-t group $A(V,C)$ is transitive on $C(E)$. If $I(V,C)$ is transitive, then (V,C) is regular. Among the 2-t ec-graphs in Theorem 15 only those listed in this theorem have transitive isometry groups, which are thus the point color symmetric ones. By definition a symmetric association scheme is a regular ec-graph. Among the 2-t ec-graphs in Theorem 15 only those listed in this theorem satisfy the definition of a symmetric association scheme. (See Bannai and Ito [5, p. 52].) □

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References

[1] M. Aschbacher, On the maximal subgroups of the finite classical groups, Invent Math. 76 (1984) 469–514.
[2] M. Aschbacher, Chromatic geometry, Quart. J. Math. Oxford Ser. 38 (2) (1987) 277–296.
[3] M. Aschbacher, Private communication, 1999.
[4] E. Artin, Geometric Algebra, Interscience Publishers, New York, 1957.
[5] E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, Menlo Park, CA, 1984.
[6] P.J. Cameron, On groups of degree n and $n-1$, and highly-symmetric edge colourings, J. London Math. Soc. 9 (2) (1975) 385–391.

[7] P.J. Cameron, SGDs with 2-t automorphism group, J. Graph Theory 32 (1999) 229–233.
[8] P.J. Cameron, G. Korchmaros, One-factorizations of complete graphs with a doubly transitive automorphism group, Bull. London Math. Soc. 25 (1993) 1–6.
[9] C.C. Chen, H.H. Teh, Constructions of point-colour-symmetric graphs, J. Combin. Theory B 27 (1979) 160–167.
[10] J.H. Conway, et al., Atlas of Finite Groups, Clarendon Press, Oxford, 1985.
[11] J.D. Dixon, B. Mortimer, Permutation Groups, Springer, New York, 1996.
[12] R.W. Hartley, Determination of the ternary linear collineation groups whose coefficients lie in the $GF(2^n)$, Ann. Math. 27 (1926) 140–158.
[13] C. Isaac, Generating two-transitive equidistance spaces, Thesis, College of St. Benedict, 1998.
[14] W.M. Kantor, Homogeneous designs and geometric lattices, J. Combin. Theory A 38 (1985) 66–74.
[15] P.B. Kleidman, The maximal subgroups of the Chevalley Groups $G_2(q)$ with q odd, the Ree groups $^2G_2(q)$, and their automorphism groups, J. Algebra 117 (1988) 30–71.
[16] P.B. Kleidman, M. Liebeck, The Subgroup Structure of the Finite Classical Groups, London Math. Society Lecture Notes, Vol. 129, Cambridge University Press, Cambridge, 1990.
[17] S.S. Magliveras, The subgroup structure of the Higman-Sims simple group, Bull. Amer. Math. Soc. 77 (4) (1971) 535–539.
[18] T.Q. Sibley, Sylow-like theorems in geometry and algebra, J. Geometry 30 (1987) 1–11.
[19] M. Suzuki, On a class of doubly transitive groups, Ann. Math. 75 (1962) 105–145.