

Kater Pendulum

Introduction

It is well-known result that the period T of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length. In principle, then, a pendulum could be used to measure g , the acceleration of gravity. However, practical difficulties—primarily in measuring the length accurately—make it unsatisfactory for high precision measurements.

It turns out that a physical pendulum—typically, a mass suspended from a knife edge—can be used for much more accurate measurements. The Kater pendulum is such a physical pendulum. It is named after its inventor, Captain Henry Kater, a captain in the British army and a Fellow of the Royal Society, who invented it around 1820. For many years, it was the standard method of measuring g to high accuracy.

The beauty of this experiment is that it allows us, using relatively simple apparatus, to measure a physical quantity very accurately indeed—we hope, to within a few parts in 10,000. It works as follows: Consider a physical pendulum with two supports that lie along a line through the center of mass, as shown in Figure 1.

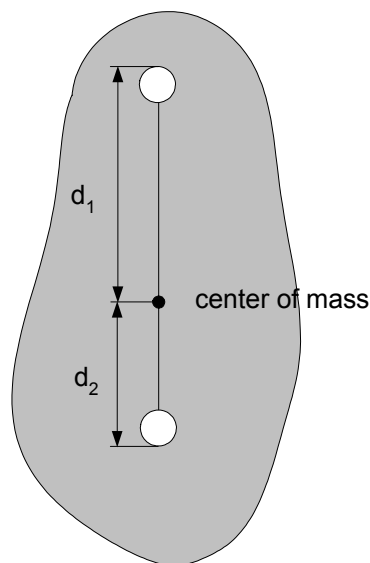


Figure 1

Kater Pendulum

The pendulum can oscillate about a knife edge through either hole. The holes are located so that the two periods of oscillation are nearly equal.

Suppose the distances d_1 and d_2 are adjusted so that the periods around both supports are equal: $T_1 = T_2 = T$. Then it can be shown that this period is the period of a simple pendulum with length $d_1 + d_2$, the distance between the supports. Thus, if the period and that distance can both be measured accurately, one can find the acceleration of gravity:

$$T = 2\pi\sqrt{\frac{d_1 + d_2}{g}} \quad \text{or} \quad g = \frac{4\pi^2(d_1 + d_2)}{T^2} \quad (1)$$

Theory

In this section, we give a detailed derivation of Equation (1). The period of a physical pendulum is

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$

where I is the moment of inertia about the axis of rotation of the pendulum. From the Parallel Axis Theorem, it is apparent that

$$I = I_c + Md^2$$

where I_c is the moment of inertia about the center of mass. Thus for our physical pendulum in Figure 1, we have

$$T_1 = 2\pi\sqrt{\frac{I_c + Md_1^2}{Mgd_1}} \quad \text{and} \quad T_2 = 2\pi\sqrt{\frac{I_c + Md_2^2}{Mgd_2}}$$

But we can always write I_c in terms of the *radius of gyration* k , the radius of a cylindrical ring that has the same mass M and the same rotational inertia I as the actual object:

$$I_c = Mk^2$$

Hence the equation for the periods can be written

$$T_1 = 2\pi\sqrt{\frac{Mk^2 + Md_1^2}{Mgd_1}} \quad \text{and} \quad T_2 = 2\pi\sqrt{\frac{Mk^2 + Md_2^2}{Mgd_2}}$$

or simplifying,

$$T_1 = 2\pi\sqrt{\frac{k^2 + d_1^2}{gd_1}} \quad \text{and} \quad T_2 = 2\pi\sqrt{\frac{k^2 + d_2^2}{gd_2}} \quad (2)$$

(Note that by using the radius of gyration, we have eliminated the mass. This step is not necessary, but it makes the rest of the derivation less cumbersome.)

If we have adjusted the distances d_1 and d_2 so that the periods are the same, it follows that

$$\frac{k^2 + d_1^2}{d_1} = \frac{k^2 + d_2^2}{d_2}$$

We bring all of the k^2 terms to the left side and rearrange as follows:

$$\begin{aligned} \frac{k^2}{d_1} + d_1 &= \frac{k^2}{d_2} + d_2 \\ k^2\left(\frac{1}{d_1} - \frac{1}{d_2}\right) &= d_2 - d_1 \\ k^2\left(\frac{d_2 - d_1}{d_2 d_1}\right) &= d_2 - d_1 \end{aligned}$$

It follows immediately that

$$k^2 = d_1 d_2$$

If we substitute this result into Equation (2), we find

$$T_1 = 2\pi\sqrt{\frac{d_1 d_2 + d_1^2}{g d_1}} \quad \text{and} \quad T_2 = 2\pi\sqrt{\frac{d_1 d_2 + d_2^2}{g d_2}}$$

Simplifying, we find

$$T_1 = T_2 = 2\pi\sqrt{\frac{d_1 + d_2}{g}},$$

the result we were seeking!

Apparatus

Our Kater pendulum consists of a 1 inch wide by ¼ inch thick brass bar, as shown in Figure 2. We have several lengths available.

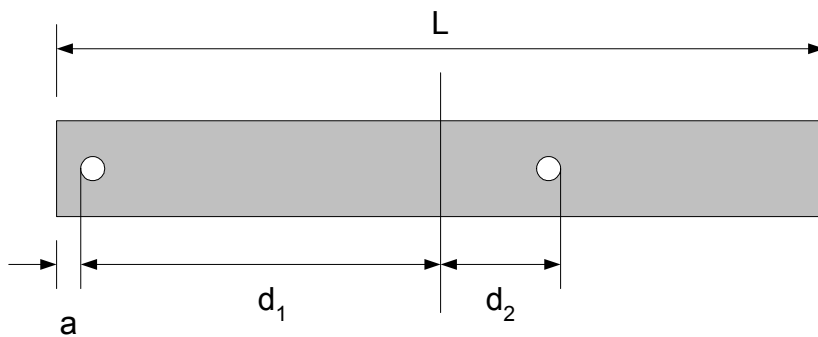


Figure 2

The vertical line marks the center of mass.

a is the distance from the end of the bar to the edge of the first hole.

There are two ways of doing the experiment. You may try either one. As you will see, it takes some time to take the data for Method 1, but the analysis is very simple. By contrast, it does not take long to take the necessary data using Method 2, but the analysis is much more complex. We aren't sure yet which method is more accurate. The experiment is new at CSB/SJU, and there are quite possibly systematic errors that we do not yet understand.

Method 1

We attach sliding weights (for the moment, large paper clips) to the bar, and measure the periods T_1 and T_2 about each hole as functions of the position of the weights, measured by the centimeter scale taped on the bar. It will turn out that at some position of the weights, the periods will be nearly equal. This position is usually someplace between the two holes. One must take some preliminary measurements to find out where that position is. Then, one takes careful measurements of period vs. weight position for several centimeters on either side of the position at which the two periods are about equal.

Thus one has two sets of data: Period vs. weight position for each support point. Do a least-squares fit to each set, and then plot both sets, with fits, on the same graph. The two fit lines will intersect each other on the graph of Period vs. weight position. That point of intersection is the

point at which the two periods are exactly equal, and one can then read that value of the period from the graph.

Once this work is done, the analysis is straightforward: From the graph, one finds the period T from the graph and then uses Equation (1) to find g . The calculation of the uncertainty is also straightforward.

Method 2

It is much easier and quicker to attach weights to the bar in such a way that T_1 is *nearly* equal to T_2 , and measure both periods accurately. However, the analysis is considerably more involved. At the cost of a little algebra, one can find an expression for g that depends on both T_1 and T_2 . We begin by squaring Equation (2) above to obtain

$$T_1^2 = \frac{4\pi^2}{g} \frac{k^2 + h_1^2}{h_1} \quad \text{and} \quad T_2^2 = \frac{4\pi^2}{g} \frac{k^2 + h_2^2}{h_2}$$

We solve each of these equations for k^2 and equate, as follows:

$$\frac{T_1^2 g d_1}{4\pi^2} - d_1^2 = k^2 = \frac{T_2^2 g d_2}{4\pi^2} - d_2^2$$

We collect all the terms involving g on the left hand side, as follows:

$$\frac{g}{4\pi^2} (T_1^2 d_1 - T_2^2 d_2) = d_1^2 - d_2^2$$

Or, rearranging, we obtain

$$\frac{4\pi^2}{g} = \frac{T_1^2 d_1 - T_2^2 d_2}{d_1^2 - d_2^2} \quad (3)$$

This result is perfectly correct. But it is not in a form that is particularly useful in this experiment, where the quantity we can measure directly is the distance between the supports, $d_1 + d_2$. For this reason, we will try to find quantities A and B that satisfy an equation of the form

$$\frac{4\pi^2}{g} = \frac{T_1^2 d_1 - T_2^2 d_2}{d_1^2 - d_2^2} = \frac{A}{d_1 + d_2} + \frac{B}{d_1 - d_2}$$

The algebra takes a few lines, and so we will work it out in Appendix 1. The result is

$$\frac{1}{g} = \frac{1}{4\pi^2} \left(\frac{T_1^2 + T_2^2}{2(d_1 + d_2)} + \frac{T_1^2 - T_2^2}{2(d_1 - d_2)} \right) \quad (4)$$

Take a moment to look at this equation, and remember that we are trying to measure g **very** accurately. It turns out that casting Equation (3) in this form allows us to do so. The first term in the brackets contains three quantities—two times, T_1 and T_2 , and the distance $d_1 + d_2$ —that we can measure very accurately. The second term contains two quantities, d_1 and d_2 , that we know much less accurately; but if T_1 and T_2 are nearly equal, the second term will be much smaller

than the first, and so the uncertainties in d_1 and d_2 will turn out to matter less. The result will be a very accurate measurement!

Procedure (both methods)

Take a few minutes to look over the apparatus. The pendulum is suspended from a knife edge mounted rigidly in a bench clamp. We measure the period using a photogate detector connected to the Pasco Science Workshop software. Your instructor will show you how this system works. Play around with it for a few minutes until you are comfortable with it.

There are a number of places where systematic error (as opposed to random error) can creep into this experiment. **In an experiment designed for high accuracy, it is important to look for and eliminate as many sources of systematic error as possible.**

It is equally important to monitor your data carefully, with rough graphs, while you are taking it. Recording a bunch of numbers, and waiting until later to see what the graphs or calculations look like, is a recipe for disaster!

Here are some things to watch out for:

1. It is important that the knife edge support be accurately level. Use a level to check. If the knife edge is not level, place small pieces of paper or foil under the knife edge or one of the clamps until you get it as nearly level as you can.
2. The pendulum should oscillate back and forth in a plane. Be sure it is not wobbling or moving from side to side. It's not a bad idea to start it oscillating, and then let it go for a few minutes, before you measure the period.
3. The plane in which pendulum oscillates should be perpendicular to the knife edge.
4. The period of a pendulum depends slightly on the amplitude of the oscillation. The equations we use, therefore, are strictly valid only in the limit of very small amplitude. Consequently, keep the amplitude as small as possible. If time permits, see if you can detect the small increase in period as the amplitude is increased.
5. The experiment is sensitive to vibration. For example, try banging on the table, or even blowing gently on the pendulum, while you measuring the period. The effect is striking. Look for ways to eliminate vibration as much as possible. For example, place the computer mouse on a different surface, so that pressing it to start the experiment will not introduce a source of vibration.

There are at least two other possibilities for systematic error: the quartz clock in the Pasco interface box that is used to measure the period, and the micrometers and calipers that we use to measure distances.

It turns out that many sources of systematic error have the effect of lengthening the period. Thus, if systematic errors are present, the value of g will tend to be systematically low.

The procedure is as follows:

For Method 1:

1. Make some initial measurements to find the point at which the periods are approximately equal.

2. For several centimeters on either side of this point, make measurements of period vs. clamp position, at about 0.5 cm to 1 cm. intervals, for both supports. Be sure to mount the clamps symmetrically on the bar. For each measurement, record the average of 6 or 8 periods and the standard deviation, and calculate the standard deviation of the mean. For later analysis, it can be convenient to record these measurements directly in the Linfit spreadsheet.
3. For each data set, make a graph and do a least-squares fit. A fit to a straight line will probably be adequate, but in some cases, the data may show enough curvature that one should fit to a quadratic. The point where the two curves intersect gives the period needed for Equation (1).
4. Measure d_1+d_2 (the distance between the two supports). It is important to make this measurement as accurately as possible. Your instructor will show you how to do these measurements accurately. Please handle the calipers and micrometers carefully!

For Method 2:

1. Make an initial measurement of the two periods, fairly roughly.
2. Mount two paper clamps on the pendulum. The idea here is to find, by trial and error, a location that makes the difference in the periods as small as possible. Be sure to mount the clamps symmetrically on the bar. A good choice to start is a position between hole 2 and the center of the bar. Once you have found an optimum position, record it in your lab notebook.
3. With the pendulum oscillating about one support, take about 100 measurements of the period. Record the average period and the standard deviation, and calculate the standard deviation of the mean.
4. Repeat for the second support.
5. Measure the distances a (the distance from the end of the bar to the first hole), d_1+d_2 (the distance between the two supports), and L (the length of the pendulum) as accurately as you can. Your instructor will show you how to do these measurements accurately. Please handle the calipers and micrometers carefully! They are sensitive, and moderately expensive.

Both methods:

1. One can set the Pasco software to stop taking data after a set elapsed time—thus, one can start timing the pendulum, and the program will stop itself, and list each reading, the number of oscillations, the mean value, and the standard deviation. Your instructor will show you how this feature of the program works.
2. Notice that the Pasco program calculates the average and the standard deviation, but does *not* calculate the standard deviation of the mean—the appropriate uncertainty for the average period! You will need to record the number of periods measured, and calculate the standard deviation of the mean. It is worth noting that the standard deviation—the uncertainty in each period—does not change appreciably, no matter how many measurements one takes.
3. It can be helpful to adjust y axis of the graph in the Pasco program so that you can see the variation in period graphically while the data are being recorded. Click on the y axis to do so.
4. The micrometer and calipers claim an accuracy of 0.001 inch (or 0.02 mm). One should, of course, regard such claims with a grain of salt, and look for ways of double-checking!

Data Analysis

Method 1

From a careful analysis of your graphs, find the point at which the periods are equal. Then, use Equation (1) to calculate g and the uncertainty in g .

Method 2

Refer to Figure 2, and see if you can calculate approximate values of d_1 and d_2 , given your measured values of L and a . These values should be measured from the center of mass. However, it turns out that the two holes do not cause the center of mass to be shifted from the center of the bar by more than a half a millimeter or so. It's easy to confirm this point by balancing the pendulum on one side, on the knife edge support. Thus, if you calculate d_1 and d_2 relative to the center of the bar instead of the center of mass, you will know $d_1 - d_2$ to an accuracy of about 1 mm. It turns out that this low accuracy measurement will not significantly affect the accuracy with which g is measured! You can then calculate $1/g$ using Equation (4).

The uncertainty analysis is a bit complicated, so we will go through it carefully in Appendix 2. The idea is to find the uncertainty in $1/g$ from Equation (4), and then use that result to find the uncertainty in g . See the Appendix for details. Be sure you understand why the large uncertainty in $d_1 - d_2$ does not affect the accuracy with which g is measured, as long as the two periods are nearly equal.

Both Methods

It is important to retain as much accuracy as possible in using either Equation (1) or Equation (4) to calculate g . If you are using a calculator, it is worth using storage registers to store intermediate results, so that no accuracy is lost in reentering numbers. You can also use a spreadsheet, or a program like Mathcad or Mathematica to do the calculations.

Conclusions

Report your value of g , with uncertainty. Discuss any systematic uncertainties or other problems with the experiment that you think need further refinement.

It might also be interesting to compare your value of g here with the value you found in the free fall experiment.

APPENDIX 1
Derivation of Equation (4)

We want to derive Equation (4) above—that is, we want to find the quantities A and B that satisfy

$$\frac{T_1^2 d_1 - T_2^2 d_2}{d_1^2 - d_2^2} = \frac{A}{d_1 + d_2} + \frac{B}{d_1 - d_2} \quad (5)$$

If we convert the right-hand side of this equation to a common denominator, we obtain

$$\frac{T_1^2 d_1 - T_2^2 d_2}{d_1^2 - d_2^2} = \frac{A(d_1 - d_2) + B(d_1 + d_2)}{d_1^2 - d_2^2}$$

Now, in the numerator of the right-hand side, collect all of the terms in d_1 and d_2 :

$$\frac{T_1^2 d_1 - T_2^2 d_2}{d_1^2 - d_2^2} = \frac{(A + B)d_1 + (B - A)d_2}{d_1^2 - d_2^2}$$

If we now compare the numerators of the left and right sides, we see at once that

$$\begin{aligned} A + B &= T_1^2 \\ B - A &= -T_2^2 \end{aligned}$$

Thus we have a pair of simultaneous equations that we can solve for A and B . One finds

$$\begin{aligned} A &= \frac{T_1^2 + T_2^2}{2} \\ B &= \frac{T_1^2 - T_2^2}{2} \end{aligned}$$

Substituting this result into Equation (5), we obtain

$$\frac{T_1^2 d_1 - T_2^2 d_2}{d_1^2 - d_2^2} = \frac{T_1^2 + T_2^2}{2(d_1 + d_2)} + \frac{T_1^2 - T_2^2}{2(d_1 - d_2)}$$

This result leads immediately to Equation (4) above.

Appendix 2 Calculation of the uncertainty in 1/g

We begin by writing Equation (4) in the form

$$\frac{1}{g} = \frac{P+Q}{4\pi^2}$$

where

$$P = \frac{T_1^2 + T_2^2}{2(d_1 + d_2)} \quad \text{and} \quad Q = \frac{T_1^2 - T_2^2}{2(d_1 - d_2)}$$

But the quantity $d_1 + d_2$ is measured as a single distance, with a single uncertainty. Likewise, one can consider $d_1 - d_2$ as a single quantity, known to within a millimeter or so. (Note that in any case, one cannot treat d_1 and d_2 as independent—as one increases, the other necessarily decreases.) Consequently, it is convenient to define

$$D \equiv d_1 + d_2 \quad \text{and} \quad d \equiv d_1 - d_2$$

Hence, our expressions for P and Q become

$$P = \frac{T_1^2 + T_2^2}{2D} \quad \text{and} \quad Q = \frac{T_1^2 - T_2^2}{2d}$$

Notice that all the quantities in P are known very accurately; however, Q is known less accurately, both because the numerator is a difference, and because d is known only to within a millimeter or so. It is for this reason that one adjusts the periods to make Q as small as possible.

The error calculation is more complicated than can easily be done using the methods in Appendix A of the Physics 191 laboratory manual. It can nevertheless be shown that

$$\delta P = P \sqrt{\left(2 \frac{T_1 \delta T_1}{T_1^2 + T_2^2}\right)^2 + \left(2 \frac{T_2 \delta T_2}{T_1^2 + T_2^2}\right)^2 + \left(\frac{\delta D}{D}\right)^2}$$

and that

$$\delta Q = Q \sqrt{\left(2 \frac{T_1 \delta T_1}{T_1^2 - T_2^2}\right)^2 + \left(2 \frac{T_2 \delta T_2}{T_1^2 - T_2^2}\right)^2 + \left(\frac{\delta d}{d}\right)^2}$$

The uncertainty in 1/g is therefore

$$\delta\left(\frac{1}{g}\right) = \frac{\sqrt{\delta P^2 + \delta Q^2}}{4\pi^2}$$

and it is an easy matter to find the uncertainty in g .

Expected Value of g at Collegeville, MN

Latitude formula: if g is measured in milligals (1 milligal = 10^{-5} m/sec²)

$$g = g_e(1 + \beta \sin^2 \theta) - 2g_e \frac{h}{R} + B$$

where

- g_e is the mean acceleration of gravity at the equator, 978,049 milligals
- θ is the latitude
- h is the altitude above sea level
- R is the radius of the earth
- β is a constant, 0.005297
- B is the Bourget anomaly

One finds g should be about 9.8059 m/sec²